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Fixed point results for multivalued contractions on graphs and their applications

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SOMMAIRE

Nous présentons dans cette thèse des théorèmes de point fixe pour des contractions multivoques définies sur des espaces métriques, et, sur des espaces de jauges munis d'un graphe. Nous illustrons également les applications de ces résultats à des inclusions intégrales et à la théorie des fractales.

Cette thèse est composée de quatre articles qui sont présentés dans quatre chapitres. Dans le chapitre 1, nous établissons des résultats de point fixe pour des fonctions multivoques, appelées G-contractions faibles. Celles-ci envoient des points connexes dans des points connexes et contractent la longueur des chemins. Les ensembles de points fixes sont étudiés. La propriété d'invariance homotopique d'existence d'un point fixe est également établie pour une famille de G-contractions multivoques faibles. Dans le chapitre 2, nous établissons l'existence de solutions pour des systèmes d'inclusions intégrales de Hammerstein sous des conditions de type de monotonie mixte. L'existence de solutions pour des systèmes d'inclusions différentielles avec conditions initiales ou conditions aux limites périodiques est également obtenue. Nos résultats s'appuient sur nos théorèmes de point fixe pour des G-contractions multivoques faibles établis au chapitre 1. Dans le chapitre 3, nous appliquons ces mêmes résultats de point fixe aux systèmes de fonctions itérées assujettis à un graphe orienté. Plus précisément, nous construisons un espace métrique muni d'un graphe G et une G-contraction appropriés. En utilisant les points fixes de cette G-contraction, nous obtenons plus d'information sur les attracteurs de ces systèmes de fonctions itérées. Dans le chapitre 4, nous considérons des contractions multivoques définies sur un espace de jauges muni d'un graphe. Nous prouvons un résultat de point fixe pour des fonctions multivoques qui envoient des points connexes dans des points connexes et qui satisfont une condition de contraction généralisée. Ensuite, nous étudions des systèmes infinis de fonctions itérées assujettis à un graphe orienté (H-IIFS). Nous donnons des conditions assurant l'existence d'un attracteur unique à un H-IIFS. Enfin, nous appliquons notre résultat de point fixe pour des contractions multivoques définies sur un espace de jauges muni d'un graphe pour obtenir plus d'information sur l'attracteur d'un H-IIFS. Plus précisément, nous construisons un espace de jauges muni d'un graphe G et une G-contraction appropriés tels que ses points fixes sont des sous-attracteurs du H-IIFS.

Mots-clés : Point fixe, contraction, fonction multivoque, inclusion intégrale de Hammerstein, inclusion différentielle, système de fonctions itérées, fractale.

SUMMARY

In this thesis, we present fixed point theorems for multivalued contractions defined on metric spaces, and, on gauge spaces endowed with directed graphs. We also illustrate the applications of these results to integral inclusions and to the theory of fractals.

This thesis is a collection of four research papers which are presented in four chapters. In Chapter 1, we establish fixed point results for the maps, called multivalued weak G-contractions, which send connected points to connected points and contract the length of paths. The fixed point sets are studied. The homotopical invariance property of having a fixed point is also established for a family of weak G-contractions. In Chapter 2, we establish the existence of solutions of systems of Hammerstein integral inclusions under mixed monotonicity type conditions. Existence of solutions to systems of differential inclusions with initial value condition or periodic boundary value condition are also obtained. Our results rely on our fixed point theorems for multivalued weak G-contractions established in Chapter 1. In Chapter 3, those fixed point results for multivalued G-contractions are applied to graph-directed iterated function systems. More precisely, we construct a suitable metric space endowed with a graph G and an appropriate G-contraction. Using the fixed points of this G-contraction, we obtain more information on the attractors of graph-directed iterated function systems. In Chapter 4, we consider multivalued maps defined on a complete gauge space endowed with a directed graph. We establish a fixed point result for maps which send connected points into connected points and satisfy a generalized contraction condition. Then, we study infinite graph-directed iterated function systems (H-IIFS). We give conditions insuring the existence of a unique attractor to an H-IIFS. Finally, we apply our fixed point result for multivalued contractions on gauge spaces endowed with a graph to obtain more information on the attractor of an H-IIFS. More precisely, we construct a suitable gauge space endowed with a graph G and a suitable multivalued G-contraction such that its fixed points are sub-attractors of the H-IIFS.

Keywords : Fixed point, contraction, multivalued map, Hammerstein integral inclusion, differential inclusion, iterated function system, fractal.

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INTRODUCTION

Over the past few decades, the theory of fixed points has become an important tool in different fields of mathematics such as best approximation theory, differential equations, operator equations and theory of fractals. The Banach contraction principle is the most versatile elementary result in fixed point theory. There is a vast amount of literature dealing with generalizations of this remarkable theorem.

In this thesis, we study fixed point results for multivalued contractions defined on a directed graph. Besides developing and improving some of the fixed point results in the literature, we also illustrate the applications of our results. In terms of applications, we focus on problems of integral inclusions, and, on the theory of fractals. This thesis is a compilation of four research articles. Each one of these articles, which were jointly authored by T. Dinevari and M. Frigon, is presented in a single chapter. Chapter 1 is devoted to the study of fixed point results for multivalued contractions defined on a complete metric space endowed with a graph. Chapter 2 deals with the applications to systems of Hammerstein integral inclusions. Chapter 3 is devoted to the applications of our fixed point results to finite graph-directed iterated function systems. Chapter 4 discusses the fixed point results for multivalued contractions defined on a complete gauge space endowed with a directed graph and their applications to infinite graph-directed iterated function systems.

0.1. Fixed points in metric spaces with graphs

In recent years, many results have been obtained extending Banach's theorem to partially ordered spaces [35, 36, 37, 40, 43, 44]. The first fixed point result on considerations of order was established by Knaster and Tarski [30, 46]. Their theorem asserts that if $(X, \preceq)$ is a complete lattice and $f : X \to X$ is order-preserving, then f has a fixed point and the set of fixed points of f is a complete lattice. In 2004, Ran and Reurings [44] proved a fixed point result which is, in some sense, a combination of the Banach contraction principle and the Knaster-Tarski theorem.

Theorem 0.1.1. *[44] Let (X,d) be a complete metric space endowed with a partial order $\preceq$ such that every pair of elements of X has an upper bound and a lower bound. Let $f : X \to X$ be monotone and continuous such that*

$$\text{there exists } \alpha \in]0,1[\text{ such that } d(f(x), f(y)) \leq \alpha d(x,y) \text{ for every } x \preceq y.$$

If there exists $x_0 \in X$ with $x_0 \preceq f(x_0)$ or $f(x_0) \succeq x_0$ then f has a unique fixed point x^* and $\lim_{n\to\infty} f^n(x) = x^*$ for every $x \in X$.

Subsequently, Nieto and Rodríguez-López [36] extended Ran and Reurings' result by replacing the continuity by an assumption ensuring that for every non-decreasing (or non-increasing) sequence $\{x_n\}$, if $x_n \to x$ then $x_n \preceq x$ (or $x \preceq x_n$) for every $n \in \mathbb{N}$. Further improvements of the above results can be found in [37, 43]. The main characteristic of these works is that the contractivity condition is only assumed to hold on comparable elements with respect to the partial order, and their main strategy involves combining the ideas of iterative methods with those of monotone methods.

In 2008, Jachymski [29] presented a nice unification of the previous results by considering graphs instead of partial orders and by introducing the notion of single-valued G-contraction in complete metric spaces endowed with a graph. Let (X,d) be a complete metric space. Consider a directed graph G such that the set $V(G)$ of its vertices coincides with X, the set $E(G)$ of its edges contains all loops, and G has no parallel edges.

Definition 0.1.1. *[29] A map $f : X \to X$ is said to be a single-valued G-contraction, if f preserves the edges of G, i.e.,*

$$\text{for all } x,y \in X \text{ if } (x,y) \in E(G) \text{ then } (f(x), f(y)) \in E(G),$$

and f decreases weights of the edges, i.e., there exists $\alpha \in]0,1[$ such that for all $x,y \in X$, if $(x,y) \in E(G)$ then $d(f(x), f(y)) \leq \alpha\, d(x,y)$.

It was proved [29] that a single-valued G-contraction $f : X \to X$, defined on a complete metric space (X,d) endowed with a graph G, has a fixed point if $X_f := \{x \in X : (x, f(x)) \in E(G)\} \neq \emptyset$ and the following property is satisfied:

$$\begin{array}{l}\text{for every sequence } \{x_n\} \text{ in } X, \text{ if } x_n \to x \text{ and } (x_n, x_{n+1}) \in E(G), \text{ then there is a} \\ \text{subsequence } \{x_{k_n}\} \text{ such that } (x_{k_n}, x) \in E(G) \text{ for every } n \in \mathbb{N}.\end{array}$$

A large amount of research following Jachymski's approach has been published during recent years, studying different aspects and applications of G-contractions.

There have been also various generalizations of the contraction principle to multivalued maps [16, 17, 20, 33]. The first one was given by Nadler [32] for multivalued contractions with nonempty closed bounded values defined on a complete metric space. A multivalued map $F : X \to X$ defined on a metric space (X, d), is said to be a contraction in the sense of Nadler, if there exists $\alpha < 1$ such that

$$D_H(F(x), F(y)) \leq \alpha\, d(x, y) \text{ for every } x, y \in X,$$

where D_H is the Hausdorff distance induced by d.

In Chapter 1 of this thesis, which is based on the paper [10] entitled *Fixed point results for multivalued contractions on a metric space with a graph*, we consider multivalued maps defined on a complete metric space endowed with a directed graph G. We generalize Jachymski's fixed point results to multivalued maps by introducing the notion of multivalued G-contraction.

Definition 0.1.2. *Let $F : X \to X$ be a multivalued map with nonempty values. We say that F is a G-contraction if there exists $\alpha \in \,]0, 1[$ such that for all $(x, y) \in E(G)$ and all $u \in F(x)$, there exists $v \in F(y)$ such that $(u, v) \in E(G)$ and $d(u, v) \leq \alpha d(x, y)$.*

This definition is more general than the one given by Nicolae, O'Regan, and Petruşel in [34], where F is said to be a multivalued contraction if

(i) there exists $\lambda \in \,]0, 1[$ such that $D(F(x), F(y)) \leq \lambda d(x, y)$ for all $(x, y) \in E(G)$;

(ii) for each $(x, y) \in E(G)$, $u \in F(x)$, and $v \in F(y)$ such that $d(u, v) \leq a d(x, y)$ for some $a \in \,]0, 1[$, one has $(u, v) \in E(G)$.

We give an affirmative answer to the question, proposed in [34], whether an existence result can be obtained by weaker contractive type conditions in the absence of condition (ii). We also extend Nadler's fixed point theorem [33] for (ε, λ)-uniformly locally contractive multivalued maps.

Subsequently, we generalize our fixed point results for G-contractions by introducing the notion of multivalued weak G-contraction. A weak G-contraction preserves the paths and decreases the weights of paths, but it does not necessarily preserve the length of paths. Thus, if F is a G-contraction, then F is a weak G-contraction; however, a weak G-contraction is not necessarily a G-contraction.

Then we compare fixed point sets of a weak G-contraction obtained by Picard iterations from different starting points.

We also establish the homotopical invariance property of having a fixed point for a family of weak G-contractions. More precisely, for $\hat{x} \in X$, $r > 0$, and $H : I \times \overline{B(\hat{x}, r)} \to X$ a multivalued map with nonempty closed values , where I is a closed subset of $[0, 1]$ such that $\{0, 1\} \subset I$, and for all $t \in I$, $H(t, \cdot) : \overline{B(\hat{x}, r)} \to X$ is a weak G-contraction, we prove that, under certain conditions, if $H(0, \cdot)$ has a fixed point then $H(1, \cdot)$ has a fixed point.

0.2. Applications to differential and integral inclusions

Fixed point theorems have been widely used to prove existence results in differential equations. In [36, 37], Nieto and Rodríguez-López applied their fixed point results on partially ordered sets that have been mentioned above, to prove the existence and uniqueness of solution for the following first order differential equation with periodic boundary conditions:

$$\begin{cases} u'(t) = f(t, u(t)), \quad t \in I = [0, T], \\ u(0) = u(T), \end{cases}$$

where $T > 0$, and $f : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function and satisfies a monotone condition.

Afterwards, Gnana Bhaskar and Lakshimikantham [23], established a fixed point result for mixed monotone operators satisfying a contractive type condition in a metric space endowed with a partial order, where a map $F : X \times X \to X$ is said to be a mixed monotone operator, if $F(x, y)$ is non-decreasing monotone in x and non-increasing monotone in y. They applied their result to periodic boundary value problems for systems of two first order differential equations under a mixed monotone condition.

In Chapter 2 of this thesis, which is based on the paper [12] entitled *Systems of Hammerstein integral inclusions in Banach spaces with mixed monotone conditions*, we establish existence results for the following system of Hammerstein integral inclusions:

$$x_i(t) \in \int_0^1 H_i(t, s, x_1(s), \dots, x_N(s))\, ds \quad \forall t \in [0, 1],\ \forall i = 1, \dots, N,$$

where, $H_i : [0, 1] \times [0, 1] \times E_1 \times \cdots \times E_N \to E_i$ is a multivalued map with nonempty values satisfying mixed monotone conditions. Here E_i is a Banach space endowed with an order for every i. Existence results for this problem have been established for $N = 1$ by many authors [3, 6, 38, 39, 41]. In our main existence result, we assume that the multivalued maps H_i are non-decreasing or non-increasing with respect to each variable x_j. The continuity type condition imposed on H_i is weaker than the notion of upper semi-continuity and the maps H_i are not needed to be closed-valued or compact-valued. We also consider the particular cases where the maps H_i are all non-increasing or are all non-decreasing. These assumptions permit us to obtain existence results with a weaker continuity type condition. Moreover, we consider the case where some of the maps H_i are non-decreasing and some are non-increasing with respect to some x_j. Our existence results are based on a slight modification of our fixed point result for multivalued weak G-contractions in Chapter 1. It is worthwhile to point out that we do not use the theory of coupled fixed point results for mixed monotone operators which is one the common methods to prove the existence results for problems with mixed monotone conditions.

0.3. Applications to finite graph-directed iterated function systems

Iterated function systems are used to generate fractals by iterating a finite collection of maps $\{T_i : i = 1, \ldots, n\}$. In 1981, Hutchinson [28] proved that if each T_i is a contraction on a complete metric space M, then there exists a unique nonempty compact set $K \subset M$ which is invariant with respect to the maps T_i; that is $K = \bigcup_{i=1}^{n} T_i(K)$. The existence of the attractor K can be deduced from the Banach contraction principle. Graph-directed constructions are natural generalizations of iterated function systems. Mauldin and Williams [31] were the firsts who introduced the notion of graph-directed constructions in $\mathbb{R}^m$ governed by a directed graph H and the similarity ratios which are labeled with the edges of the graph. Indeed, a graph-directed construction in $\mathbb{R}^m$ consists of a finite sequence of non overlapping compact subsets of $\mathbb{R}^m$: $J_1, \ldots, J_n$ such that each J_i has a nonempty interior, a directed graph H with $V(H) = \{1, \ldots, n\}$, and for each $(i, j) \in E(H)$, a similarity map $T_{i,j} : \mathbb{R}^m \to \mathbb{R}^m$, such that

$$\bigcup\{T_{i,j}(J_j) : (i, j) \in E(H)\} \subseteq J_i.$$

It was proved that each graph-directed construction has a unique attractor [31]. Graph-directed constructions have been studied and generalized by many authors [8, 9, 14, 15]. In particular, it was shown that with an appropriate rescaling, similarities can be replaced by contractions [8].

In Chapter 3 of this thesis, which is based on the paper [11] entitled *Applications of multivalued contractions on graphs to graph-directed iterated function systems*, we apply our fixed point result for multivalued G-contractions to finite graph-directed iterated function systems. We consider a directed graph $H = (V(H), E(H))$ such that $V(H) = \{1, \ldots, n\}$, H has no parallel edges, and *outdeg*$(i) \geq 1$ for every $i \in V(H)$. A graph-directed iterated function system over the graph H is a collection of n nonempty, bounded, complete metric spaces, $(X_1, d_1), \ldots, (X_n, d_n)$, and, for each $(i, j) \in E(H)$, a contraction $T_{i,j} : X_j \to X_i$ with constant of contraction $\lambda_{i,j}$. Using the graph H and the metric spaces X_i, we define a complete metric space X endowed with a suitable directed graph G.

Then we construct an appropriate multivalued G-contraction on X. Using the fixed points of this G-contraction, we obtain more information on the attractors of graph-directed iterated function systems by considering the set of connected components and the set of connecting paths of H. Indeed, we study certain subsets of the attractor and the relations between these subsets.

0.4. Fixed points in gauge spaces with graphs and applications to infinite graph-directed iterated function systems

The Banach contraction principle has been extended to gauge spaces in different ways. In 1974, Tarafdar [45] generalized Banach's theorem to complete gauge spaces by introducing the notion of contraction in complete gauge spaces. A single-valued map f defined on a complete gauge space $(X, \{q_s\}_{s\in S})$ is said to be a contraction in the sense of Tarafdar, if for all $s \in S$ there exists $k_s < 1$ such that $q_s\big(f(x), f(y)\big) \leq k_s q_s(x,y)$ for all $x, y \in X$. Subsequently, Gheorghiu [22] established a fixed point result which was generalized by Chiş and Precup [5] as follows.

Theorem 0.4.1 ([5]). *Let $(X, \{q_s\}_{s\in S})$ be a complete gauge space, and $f : X \to X$ be a single-valued map. Assume that there exist a function $\psi : S \to S$ and $k = (k_s)_{s\in S}$ such that $k_s \geq 0$ for all $s \in S$, and*

$$q_s\big(f(x), f(y)\big) \leq k_s q_{\psi(s)}(x,y) \quad \forall s \in S,\ \forall x, y \in X, \tag{0.4.1}$$

and

$$\sum_{n=1}^{\infty} k_s k_{\psi(s)} \cdots k_{\psi^{n-1}(s)} q_{\psi^n(s)}(x,y) < \infty \quad \forall s \in S,\ \forall x, y \in X, \tag{0.4.2}$$

where ψ^n is the n-th iteration of ψ. Suppose that for every $x_0 \in X$, if $\{f^n(x_0)\}$ converges to some $x \in X$, then $x = f(x)$. Then f has a unique fixed point.

In Chapter 4 of this thesis, which is based on the paper [13] entitled *A contraction principle on gauge spaces with graphs and applications to infinite graph-directed iterated function systems*, we consider multivalued maps defined on a complete gauge space endowed with a directed graph. Let $(X, \{q_s\}_{s\in S})$ be a complete gauge space endowed with a graph G. We define the notion of multivalued contraction on complete gauge spaces endowed with a directed graph. That is a multivalued map preserving the edges as in the Definition 0.1.2, and satisfying the contraction condition in the sense of Gheorghiu (0.4.1).

We generalize Theorem 0.4.1 to multivalued G-contractions defined on a complete gauge space with a graph. In the case where X is a metric space, this result generalizes our fixed point theorem presented in Chapter 1.

We also study infinite graph-directed iterated function systems. We consider a directed graph H with countable vertices, such that H has no parallel edges and $1 \leq outdeg(i) < \infty$ for all $i \in V(H)$. We introduce the notion of *infinite graph directed function system* over the graph H (H-IIFS). Therefore the corresponding graph iterated function system $\{T_{i,j}\}_H$, includes infinite number of contractions. We give conditions insuring the existence of an attractor to an H-IIFS. Then we study certain subsets of the attractor and the relations between these sub-attractors. To this aim, we construct a gauge space endowed with a graph G, and a suitable multivalued G-contraction. Using our fixed point result for multivalued G-contractions in gauge spaces, we obtain the sub-attractors of the H-IIFS as the fixed points of the constructed G-contraction.

Chapter 1

FIXED POINT RESULTS FOR MULTIVALUED CONTRACTIONS ON A METRIC SPACE WITH A GRAPH

1.1. Introduction and preliminaries

In 2004, Ran and Reurings [44] were the first who had obtained a fixed point result which is, in some sense, a combination of the Banach contraction principle and the Knaster-Tarski fixed point theorem in a partially ordered set. Indeed, Ran and Reurings considered a monotone, order preserving single-valued map f defined on a complete metric space endowed with a partial ordering. They assumed that f satisfies a contraction condition not necessarily for all x and y, but for those such that $x \leq y$. Subsequently, Nieto and Rodríguez-López [36] proved a modified version of Ran and Reurings' result by replacing the continuity assumption by a condition insuring that $x_n \leq x$ for all n, as soon as $(x_n)_{n\in\mathbb{N}}$ is an increasing sequence converging to x. Later, their results were generalized to maps not necessarily monotone but which are such that the images of comparable points are comparable [37, 43]. Their results were also extended to L-spaces [35, 43].

In 2008, Jachymski [29] presented a nice unification of most of the previous results by considering complete metric spaces endowed with a graph G. He introduced the notion of single-valued G-contraction for which he obtained fixed point results. He also compared the cardinality of the fixed point set with the cardinality of weakly connected subgraphs of G.

In this paper, we extend some fixed point results of Jachymski [29] to multivalued maps. To this aim, we introduce the notions of multivalued G-contractions and weak G-contractions for which we establish fixed point theorems. We extend also results of Nicolae, O'Regan, and Petruşel [34] for multivalued maps. Since our maps are multivalued, it is not possible to compare the cardinality of the fixed point set with the cardinality of connected subgraphs. However, we present a comparison between fixed point sets obtained from Picard iterations starting from different points.

We also obtain a homotopical invariance result for a family of multivalued weak G-contractions. This result is new even for single-valued contractions on a complete metric space endowed with a partial ordering or a graph. It generalizes a result in [18] for families of classical multivalued contractions.

Finally, we extend our notions of contractions to multivalued (G,φ)-contractions for which we obtain fixed point results; here φ is a strong comparison function. Let us mention that different notions of single-valued φ-contractions using comparison functions were introduced and fixed point results were obtained in complete metric spaces endowed with a partial ordering or a directed graph in particular in [1, 4, 34, 40].

First, we introduce some notations.

Let (X,d) be a complete metric space. For a directed graph G, the set of its vertices and the set of its edges are denoted by $V(G)$ and $E(G)$ respectively. We assume that $X=V(G)$, Δ the diagonal in $X\times X$ is contained in $E(G)$, and G has no parallel edges. We identify G with the pair $(V(G),E(G))$.

We denote by G_x, the subgraph of G consisting of all edges and vertices which are contained in some directed path beginning at x. In particular, for $N\in\mathbb{N}$, we say that $(x^i)_{i=0}^N$ is an N-*directed path* from x to y if $x=x^0$, $y=x^N$, and $(x^{i-1},x^i)\in E(G)$ for every $i=1,\dots,N$. We denote

$$
\begin{aligned}
[x]_G^N &:= \{y\in X:\ \text{there is an } N\text{-directed path from } x \text{ to } y\},\\
[x]_G &:= \bigcup_{N\in\mathbb{N}} [x]_G^N.
\end{aligned}
$$

So, $[x]_G=V(G_x)$. Observe that $[x]_G^1\subset[x]_G^2\subset\cdots\subset[x]_G$ since $\Delta\subset E(G)$.

For $y\in[x]_G^N$ and $z\in[x]_G$, we define

$$
p_N(x,y):=\inf\left\{\sum_{i=1}^N d(x^{i-1},x^i) : (x^i)_{i=0}^N \text{ is an } N\text{-directed path from } x \text{ to } y\right\};
$$

$$
p(x,z):=\inf\left\{\sum_{i=1}^N d(x^{i-1},x^i) : (x^i)_{i=0}^N \text{ is an } N\text{-directed path from } x \text{ to } z \text{ for some } N\in\mathbb{N}\right\}.
$$

Notice that $p_N(x,y)\ge p_{N+m}(x,y)$ for all $m\in\mathbb{N}$, and

$$
p(x,z)=\inf\{p_N(x,z) : N\in\mathbb{N} \text{ such that } z\in[x]_G^N\}
$$

since $\Delta\subset E(G)$.

A subgraph $\hat{G}$ of G is *connected* if there is a path between any two vertices of $\hat{G}$ lying in $\hat{G}$, i.e. $[x]_{\hat{G}} = [y]_{\hat{G}}$ for all $x, y \in V(\hat{G})$.

1.2. Fixed point results for G-contractions

In this section, we introduce a notion of multivalued contraction with respect to the graph G for which we establish fixed point results.

Definition 1.2.1. *Let $F : X \to X$ be a multivalued map with nonempty values. We say that F is a G-contraction if there exists $\alpha \in]0, 1[$ such that*

(C_G) for all $(x, y) \in E(G)$ and all $u \in F(x)$, there exists $v \in F(y)$ such that $(u, v) \in E(G)$ and $d(u, v) \leq \alpha d(x, y)$.

It is worthwhile to point out that a G-contraction does not need to have closed values.

Remark 1.2.1. *Let us recall that $F : X \to X$ a multivalued mapping with nonempty closed values is a contraction in the classical sense (see Covitz and Nadler [7]) if*

(C) there exists $\lambda \in]0, 1[$ such that $D(F(x), F(y)) \leq \lambda d(x, y)$ for all $x, y \in X$, where D is the generalized Hausdorff distance:

$$D(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right\} \in [0, \infty].$$

Of course, a multivalued contraction is a G-contraction (with $E(G) = X \times X$) but the inverse is false.

Remark 1.2.2. *Nicolae, O'Regan, and Petruşel [34] extended the notion of multivalued contraction on a metric space with a graph in considering the following conditions:*

(i) there exists $\lambda \in]0, 1[$ such that $D(F(x), F(y)) \leq \lambda d(x, y)$ for all $(x, y) \in E(G)$;

(ii) for each $(x, y) \in E(G)$, $u \in F(x)$, and $v \in F(y)$ such that $d(u, v) \leq ad(x, y)$ for some $a \in]0, 1[$, one has $(u, v) \in E(G)$.

It is worth noticing that if conditions (i) and (ii) hold, then (C_G) is satisfied with any $\alpha \in]\lambda, 1[$. On the other hand, (C_G) implies that for all $(x, y) \in E(G)$,

$$D_1\Big(F(x), F(y)\Big) \leq \alpha d(x, y),$$

where for $A, B \subset X$, $D_1(A, B) := \sup\{d(a, B) : a \in A\}$. However, in general, ($C_G$) does not imply (ii) as shown in Example 1.2.1, and in Example 1.2.2 for an undirected graph G. Therefore, our definition of G-contraction is more general than the definition of multivalued contraction in the sense of Nicolae, O'Regan, and Petruşel [34] even if G is undirected.

Example 1.2.1. *Let* $X = \{0\} \cup \{\frac{1}{2^n} : n \in \mathbb{N} \cup \{0\}\}$. *Consider the directed graph* G *such that* $V(G) = X$ *and*

$$E(G) = \left\{ \left(\frac{1}{2^n}, \frac{1}{2^{n+1}} \right), \left(\frac{1}{2^n}, 0 \right) : n \in \mathbb{N} \cup \{0\} \right\} \cup \Delta.$$

Let $F : X \to X$ *be defined by*

$$F(x) = \begin{cases} \left\{0, \frac{1}{2}, 1\right\} & \text{if } x = 0, \\ \left\{\frac{1}{2^{n+1}}, 1\right\} & \text{if } x = \frac{1}{2^n},\ n \in \mathbb{N}, \\ \left\{\frac{1}{2}\right\} & \text{if } x = 1. \end{cases}$$

Then F *is a multivalued* G*-contraction with constant* $\alpha = \frac{1}{2}$. *However,* F *is not a multivalued contraction in the classical sense and it is not a multivalued contraction in the sense of Nicolae, O'Regan, and Petruşel. Indeed,*

$$D\Big(F\Big(\frac{1}{2^n}\Big), F\Big(0\Big) \Big) > d\Big(0, \frac{1}{2^n}\Big) \quad \forall n \geq 2.$$

Observe also that condition (ii) of the previous remark is not satisfied. Indeed,

$$(1,0) \in E(G),\ \frac{1}{2} \in F(1),\ 1 \in F(0),\ d\Big(\frac{1}{2}, 1\Big) < d(1,0), \quad \text{but } \Big(\frac{1}{2}, 1\Big) \notin E(G).$$

Example 1.2.2. *Let* $X = \{0\} \cup \{\frac{1}{2^n} : n \in \mathbb{N} \cup \{0\}\}$. *Consider the undirected graph* G *such that* $V(G) = X$ *and*

$$E(G) = \left\{ \left(\frac{1}{2^n}, \frac{1}{2^{n+2}} \right), \left(\frac{1}{2^{n+2}}, \frac{1}{2^n} \right), \left(\frac{1}{2^n}, 0 \right), \left(0, \frac{1}{2^n} \right) : n \in \mathbb{N} \cup \{0\} \right\} \cup \Delta.$$

Let $F : X \to X$ *be defined by*

$$F(x) = \begin{cases} \{0\} & \text{if } x = 0, \\ \left\{\frac{1}{2^{n+1}}, \frac{1}{2^{n+2}}\right\} & \text{if } x = \frac{1}{2^n}, n \in \mathbb{N} \cup \{0\}. \end{cases}$$

Then F *is a multivalued* G*-contraction with constant* $\alpha = \frac{1}{2}$. *However,* F *is not a multivalued contraction in the sense of Nicolae, O'Regan, and Petruşel. Indeed, condition (ii) of Remark 1.2.2 is not satisfied since*

$$\Big(1, \frac{1}{4}\Big) \in E(G),\ \frac{1}{4} \in F(1),\ \frac{1}{8} \in F\Big(\frac{1}{4}\Big),\ d\Big(\frac{1}{4}, \frac{1}{8}\Big) < d\Big(1, \frac{1}{4}\Big),\ \text{but } \Big(\frac{1}{4}, \frac{1}{8}\Big) \notin E(G).$$

Here is a lemma which will be useful to establish our fixed point results.

Lemma 1.2.1. *Let* $F : X \to X$ *be a multivalued* G*-contraction with constant* α*, and let* $\varepsilon > 0$ *and* $N \in \mathbb{N}$. *Then for every* $x \in X$ *and* $y \in [x]_G^N$*, one has*

for any $x_1 \in F(x)$*, there exists* $y_1 \in F(y) \cap [x_1]_G^N$ *such that*

$$p_N(x_1, y_1) \leq \alpha\Big(p_N(x, y) + \varepsilon\Big); \quad (1.2.1)$$

and inductively for all $k \in \mathbb{N}$,

$$\text{for any } x_{k+1} \in F(x_k), \text{ there exists } y_{k+1} \in F(y_k) \cap [x_{k+1}]_G^N \text{ such that}$$
$$p_N(x_{k+1}, y_{k+1}) \leq \alpha^{k+1}\Big(p_N(x,y) + \varepsilon\Big). \quad (1.2.2)$$

Proof. Let $\left(x^i\right)_{i=0}^N$ be an N-directed path from x to y such that

$$\sum_{i=1}^{N} d(x^{i-1}, x^i) \leq p_N(x,y) + \varepsilon.$$

Since F is a multivalued G-contraction, for any $x_1 \in F(x)$, there exists $x_1^1 \in F(x^1)$ such that

$$(x_1, x_1^1) \in E(G) \quad \text{and} \quad d(x_1, x_1^1) \leq \alpha d(x, x^1);$$

and recursively from $i = 2$ to N, there exists $x_1^i \in F(x^i)$ such that

$$(x_1^{i-1}, x_1^i) \in E(G) \quad \text{and} \quad d(x_1^{i-1}, x_1^i) \leq \alpha d(x^{i-1}, x^i).$$

Hence, if we denote $y_1 = x_1^N$, we get

$$p_N(x_1, y_1) \leq \sum_{i=1}^{N} d(x_1^{i-1}, x_1^i) \leq \alpha \sum_{i=1}^{N} d(x^{i-1}, x^i) \leq \alpha\Big(p_N(x,y) + \varepsilon\Big).$$

Now, inductively for $k \geq 1$ and for all $x_{k+1} = x_{k+1}^0 \in F(x_k)$, from $i = 1$ to N, there exists $x_{k+1}^i \in F(x_k^i)$ such that

$$(x_{k+1}^{i-1}, x_{k+1}^i) \in E(G) \quad \text{and} \quad d(x_{k+1}^{i-1}, x_{k+1}^i) \leq \alpha d(x_k^{i-1}, x_k^i).$$

Set $y_{k+1} = x_{k+1}^N$, one has

$$p_N(x_{k+1}, y_{k+1}) \leq \sum_{i=1}^{N} d(x_{k+1}^{i-1}, x_{k+1}^i) \leq \alpha \sum_{i=1}^{N} d(x_k^{i-1}, x_k^i) \leq \alpha^{k+1} \sum_{i=1}^{N} d(x^{i-1}, x^i)$$
$$\leq \alpha^{k+1}\Big(p_N(x,y) + \varepsilon\Big).$$

□

Definition 1.2.2. *Let $F : X \to X$ be a multivalued mapping.*

(1) *Let $N \in \mathbb{N}$. We say that a sequence $(x_n)_{n\in\mathbb{N}}$ is a G_N-Picard trajectory from x_0 if $x_n \in [x_{n-1}]_G^N \cap F(x_{n-1})$ for all $n \in \mathbb{N}$. We denote by $T_N(F, G, x_0)$, the set of all such G_N-Picard trajectories from x_0.*

(2) *We say that a sequence $(x_n)_{n\in\mathbb{N}}$ is a G-Picard trajectory from x_0 if $x_n \in [x_{n-1}]_G \cap F(x_{n-1})$ for all $n \in \mathbb{N}$. We denote by $T(F, G, x_0)$, the set of all such G-Picard trajectories from x_0.*

Definition 1.2.3. *Let $F : X \to X$ be a multivalued mapping.*

(1) Let $N \in \mathbb{N}$. We say that F is G_N-Picard continuous from x_0 if the limit of any convergent sequence $(x_n)_{n\in\mathbb{N}} \in T_N(F, G, x_0)$ is a fixed point of F.

(2) We say that F is G-Picard continuous from x_0 if the limit of any convergent sequence $(x_n)_{n\in\mathbb{N}} \in T(F, G, x_0)$ is a fixed point of F.

Remark 1.2.3. *If F is G-Picard continuous from x_0 then it is G_N-Picard continuous from x_0 for every $N \in \mathbb{N}$. Observe that if F has a closed graph then F is G-Picard continuous from x_0.*

Now, we can establish a fixed point result for G-contractions.

Theorem 1.2.1. *Let $F : X \to X$ be a multivalued G-contraction. Assume there is $N \in \mathbb{N}$ such that*

(i)$_N$ there exists $x_0 \in X$ such that $[x_0]_G^N \cap F(x_0) \neq \emptyset$;

(ii)$_N$ F is G_N-Picard continuous from x_0.

Then there exists a G_N-Picard trajectory $(x_n)_{n\in\mathbb{N}}$ converging to x a fixed point of F.

Proof. Let $x_1 \in [x_0]_G^N \cap F(x_0)$. By Lemma 1.2.1, for every $\varepsilon > 0$, there exists $x_2 \in F(x_1) \cap [x_1]_G^N$ such that

$$d(x_1, x_2) \leq p_N(x_1, x_2) \leq \alpha\Big(p_N(x_0, x_1) + \varepsilon\Big)$$

and, for $n \geq 2$, there exists $x_{n+1} \in F(x_n) \cap [x_n]_G^N$ such that

$$d(x_n, x_{n+1}) \leq p_N(x_n, x_{n+1}) \leq \alpha^n\Big(p_N(x_0, x_1) + \varepsilon\Big).$$

Thus, $(x_n)_{n\in\mathbb{N}}$ is a G_N-Picard trajectory and, for $m \geq 1$,

$$d(x_n, x_{n+m}) \leq \sum_{i=0}^{m-1} d(x_{n+i}, x_{n+i+1}) \leq \sum_{i=0}^{m-1} \alpha^{n+i}\Big(p_N(x_0, x_1) + \varepsilon\Big) \leq \frac{\alpha^n\Big(p_N(x_0, x_1) + \varepsilon\Big)}{1-\alpha}.$$

So, $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Since F is G_N-Picard continuous from x_0, x the limit of $(x_n)_{n\in\mathbb{N}}$ is a fixed point of F. □

Corollary 1.2.1. *Let $F : X \to X$ be a multivalued G-contraction with closed values. Assume there is $N \in \mathbb{N}$ such that* (i)$_N$ and the following property is satisfied:

(iii)$_N$ for every $(x_n)_{n\in\mathbb{N}}$ in $T_N(F, G, x_0)$ such that $x_n \to x$, there exists $(n_k)_{k\in\mathbb{N}}$ such that $(x_{n_k}, x) \in E(G)$ for $k \in \mathbb{N}$.

Then there exists a G_N-Picard trajectory $(x_n)_{n\in\mathbb{N}}$ converging to x a fixed point of F.

Proof. Let $(x_n)_{n\in\mathbb{N}}$ in $T_N(F,G,x_0)$ be such that $x_n \to x$. By (iii)$_N$, there exists $(x_{n_k}, x) \in E(G)$ for $k \in \mathbb{N}$. Since F is a multivalued G-contraction, for all $k \in \mathbb{N}$, there exists $y_{n_k+1} \in F(x)$ such that $(x_{n_k+1}, y_{n_k+1}) \in E(G)$ and $d(x_{n_k+1}, y_{n_k+1}) \leq \alpha d(x_{n_k}, x)$. Thus,

$$d(y_{n_k+1}, x) \leq d(y_{n_k+1}, x_{n_k+1}) + d(x_{n_k+1}, x) \leq \alpha d(x_{n_k}, x) + d(x_{n_k+1}, x).$$

Therefore, $y_{n_k+1} \to x$ and hence, $x \in F(x)$, since $F(x)$ is closed. So, F is G_N-Picard continuous. The conclusion follows from the previous theorem. □

The Nadler fixed point theorem for multivalued contractions is a corollary of Corollary 1.2.1 with $E(G) = X \times X$ and $N = 1$.

Theorem 1.2.1 generalizes a result due to Jachymski [29] for single-valued maps.

Corollary 1.2.2. *Let $f : X \to X$ be a single valued G-contraction such that for some $x_0 \in X$, $f(x_0) \in [x_0]_G$. Assume that one of the following conditions holds:*

(i) if $f^n(x_0) \to x$ then $x = f(x)$;

(ii) if $f^n(x_0) \to x$ then there exists a subsequence $\left(f^{n_k}(x_0)\right)_{k\in\mathbb{N}}$ such that $(f^{n_k}(x_0), x) \in E(G)$ for $k \in \mathbb{N}$.

Then f has a fixed point.

Corollary 1.2.3 (Jachymski [29])**.** *Let $f : X \to X$ be a single valued G-contraction. Assume that one of the following conditions holds:*

(i) for all $x, y \in X$ and any sequence $(n_k)_{k\in\mathbb{N}}$ of positive integers such that $f^{n_k}(x) \to y$ and $(f^{n_k}(x), f^{n_{k+1}}(x)) \in E(G)$ for all $k \in \mathbb{N}$, one has

$$f\Big(f^{n_k}(x)\Big) \to f(y);$$

(ii) for any $(x_n)_{n\in\mathbb{N}}$ such that $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbb{N}$, then there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ with $(x_{n_k}, x) \in E(G)$ for $k \in \mathbb{N}$.

If there exists $x_0 \in X$ such that $(x_0, f(x_0)) \in E(G)$, then f has a fixed point.

Theorem 1.2.1 generalizes also a result due to Nicolae, O'Regan and Petruşel [34] for multivalued maps in the case where $N = 1$.

Corollary 1.2.4 (Nicolae, O'Regan and Petruşel [34])**.** *Let $F : X \to X$ be a multivalued map with nonempty closed values. Assume that*

(i) there exists $\lambda \in]0,1[$ such that

$$D(F(x), F(y)) \leq \lambda d(x,y) \quad \forall (x,y) \in E(G);$$

(ii) for each $(x,y) \in E(G)$, each $u \in F(x)$, $v \in F(y)$ satisfying $d(u,v) \leq ad(x,y)$ for some $a \in]0,1[$, we have $(u,v) \in E(G)$;

(iii) for any $(x_n)_{n\in\mathbb{N}}$ such that $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbb{N}$, then there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ with $(x_{n_k}, x) \in E(G)$ for $k \in \mathbb{N}$.

If there exist $x_0, x_1 \in X$ such that $x_1 \in [x_0]_G^1 \cap F(x_0)$, then F has a fixed point.

Remark 1.2.4. *In Remark 4.3 in [34], Nicolae, O'Regan and Petruşel asked the question if an existence result can be obtained in the absence of condition (ii). Theorem 1.2.1 gives a positive answer to their question.*

In Theorem 1.2.1, we obtained the existence of a G_N-Picard trajectory converging to a fixed point of F. Observe that, for $M < N$, $T_M(G, F, x_0) \subset T_N(F, G, x_0)$, since $[x]_G^M \subset [x]_G^N$. Therefore, if F is G_N-Picard continuous, then F is G_M-Picard continuous. In other words, assumption $(\text{i})_N$ is weaker than $(\text{i})_M$, while assumption $(\text{ii})_N$ is stronger than $(\text{ii})_M$. Similarly, assumption $(\text{iii})_N$ is stronger than $(\text{iii})_M$. From these observations, we obtain the following result.

Theorem 1.2.2. *Let $F : X \to X$ be a multivalued G-contraction and let x_0 be such that $[x_0]_G \cap F(x_0) \neq \emptyset$. Assume one of the following conditions hold:*

(i) F is G-Picard continuous from x_0;

(ii) F has closed values and, for every $(x_n)_{n\in\mathbb{N}}$ in $T(F, G, x_0)$ such that $x_n \to x$, there exists $(n_k)_{k\in\mathbb{N}}$ such that $(x_{n_k}, x) \in E(G)$ for $k \in \mathbb{N}$.

Then there exist $N \in \mathbb{N}$ and a G_N-Picard trajectory $(x_n)_{n\in\mathbb{N}}$ converging to x a fixed point of F.

The previous theorem can be used to obtain a generalization of the fixed point theorem for (ε, λ)-uniformly locally contractive multivalued maps due to Nadler [32].

Corollary 1.2.5. *Let $F : X \to X$ be a multivalued map with closed nonempty values. Assume there exists $\varepsilon > 0$ such that*

(i) there exists $\lambda \in \,]0, 1[$ such that F is (ε, λ)-uniformly locally contractive, i.e. for every $x, y \in X$ such that $d(x, y) < \varepsilon$, one has $D(F(x), F(y)) \leq \lambda d(x, y)$;

(ii) there exist x_0 and $\hat{x} \in F(x_0)$ such that $\{x_0, \hat{x}\}$ is ε-chainable in X; i.e. there is a finite set of points $\{x_0, \dots, x_N\} \subset X$ such that $x_N = \hat{x}$, and $d(x_{i-1}, x_i) < \varepsilon$ for all $i = 1, \dots, N$.

Then F has a fixed point.

Proof. Consider the graph G with $V(G) = X$ and

$$E(G) = \{(x, y) \in X \times X : d(x, y) < \varepsilon\}.$$

It is easy to verify that F is a multivalued G-contraction with a constant $\alpha \in \,]\lambda, 1[$. By (ii), $\hat{x} \in F(x_0) \cap [x_0]_G$. The conclusion follows from Theorem 1.2.2. □

Remark 1.2.5. *In Nadler's result for (ε, λ)-uniformly locally contractive multivalued maps, it is assumed that X is ε-chainable; i.e. $\{x, y\}$ is ε-chainable in X for every $x, y \in X$.*

1.3. Weak G-contractions

In this section, we generalize the notion of G-contraction. We present fixed point results for such generalized G-contraction.

Definition 1.3.1. *Let $Y \subset X$. We say that $F : Y \to X$, a multivalued mapping with nonempty values, is a weak G-contraction if there exists $\alpha \in \,]0, 1[$ such that*
(wC_G) for all $x, y \in Y$ with $y \in [x]_G$, and all $u \in F(x)$, there exists $v \in F(y)$ such that $v \in [u]_G$ and $p(u, v) \leq \alpha p(x, y)$.

Remark 1.3.1. *If F is a G-contraction, then F is a weak G-contraction. Indeed, for $x \in X$ and $y \in [x]_G$, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $y \in [x]_G^N$ and*

$$p_N(x, y) \leq p(x, y) + \varepsilon.$$

By Lemma 1.2.1, for $u \in F(x)$, there exists $v \in [u]_G^N \cap F(y)$ such that

$$p_N(u, v) \leq \alpha\Big(p_N(x, y) + \varepsilon\Big) \leq \alpha\Big(p(x, y) + 2\varepsilon\Big).$$

So, $v \in [u]_G$ and

$$p(u, v) \leq \alpha(p(x, y) + 2\varepsilon).$$

Since ε is arbitrary, we get the conclusion.

Observe that a weak G-contraction F is not necessarily a G-contraction. Moreover, if there is an N-directed path from x to y, there may be no N-directed path from $u \in F(x)$ to any elements of $F(y)$.

Example 1.3.1. *Let $X = \{0\} \cup \{\frac{1}{2^n} : n \in \mathbb{N} \cup \{0\}\}$. Consider the graph G such that $V(G) = X$ and*

$$E(G) = \left\{\left(\frac{1}{2^n}, \frac{1}{2^{n+1}}\right) : n \in \mathbb{N} \cup \{0\}\right\} \cup \left\{\left(\frac{1}{2^{n-1}}, 0\right) : n \text{ odd}\right\} \cup \Delta.$$

Let $F : X \to X$ be defined by

$$F(x) = \begin{cases} \left\{0, \frac{1}{2}, 1\right\} & \text{if } x = 0, \\ \left\{\frac{1}{2^{n+1}}, 1\right\} & \text{if } x = \frac{1}{2^n} \text{ with } n \text{ even}, \\ \left\{\frac{1}{2^{n+2}}, 1\right\} & \text{if } x = \frac{1}{2^n} \text{ with } n \text{ odd}, \\ \left\{\frac{1}{2}\right\} & \text{if } x = 1. \end{cases}$$

Then F is a multivalued weak G-contraction with constant $\alpha = 3/4$, but F is not a G-contraction. Indeed, for $x = \frac{1}{2^2}$, $y = \frac{1}{2^3}$, and $u = \frac{1}{2^3} \in F(x)$, there is no $v \in F(y)$ such that $(\frac{1}{2^3}, v) \in E(G)$.

We obtain a fixed point result for weak G-contractions.

Theorem 1.3.1. *Let $F : X \to X$ be a multivalued weak G-contraction. Assume there exists $x_0 \in X$ such that $[x_0]_G \cap F(x_0) \neq \emptyset$, and one of the following conditions holds:*

(i) F is G-Picard continuous from x_0;

(ii) F has closed values and, for every $(x_n)_{n\in\mathbb{N}}$ in $T(F, G, x_0)$ such that $x_n \to x$, there exists $(n_k)_{k\in\mathbb{N}}$ such that $p(x_{n_k}, x) \to 0$ and $x \in [x_{n_k}]_G$ for $k \in \mathbb{N}$.

Then there exists a G-Picard trajectory $(x_n)_{n\in\mathbb{N}}$ converging to x a fixed point of F.

Proof. Let $x_1 \in [x_0]_G \cap F(x_0)$. Since F is a weak G-contraction, there exists $(x_n)_{n\in\mathbb{N}}$ a G-Picard trajectory such that

$$p(x_n, x_{n+1}) \leq \alpha p(x_{n-1}, x_n) \leq \alpha^n p(x_0, x_1) \quad \forall n \in \mathbb{N}.$$

From the facts that $\alpha < 1$ and $d(x_n, x_{n+1}) \leq p(x_n, x_{n+1})$, we deduce that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence which converges to some $x \in X$.

If (i) is satisfied, then x is a fixed point of F.

On the other hand, if (ii) is satisfied, there exists $(n_k)_{k\in\mathbb{N}}$ such that $x \in [x_{n_k}]_G$. Since F is a weak G-contraction, there exists $y_{n_k} \in [x]_G \cap F(x)$ such that

$$p(x_{n_k+1}, y_{n_k+1}) \leq \alpha p(x_{n_k}, x) \quad \forall k \in \mathbb{N}.$$

So,

$$\begin{aligned} d(x, y_{n_k+1}) \leq d(x, x_{n_k+1}) + d(x_{n_k+1}, y_{n_k+1}) &\leq d(x, x_{n_k+1}) + p(x_{n_k+1}, y_{n_k+1}) \\ &\leq d(x, x_{n_k+1}) + \alpha p(x_{n_k}, x) \to 0. \end{aligned}$$

Therefore, $x \in F(x)$, since the values of F are closed. □

Observe that Theorem 1.2.2 is a corollary of the previous theorem. For single-valued maps, we obtain the following corollary which generalizes Corollary 1.2.2.

Corollary 1.3.1. *Let $f : X \to X$ be a single valued map such that for some $x_0 \in X$, $f(x_0) \in [x_0]_G$. Suppose that there exists $\alpha \in]0, 1[$ such that*

$$\text{for any } x, y \in X \text{ with } y \in [x]_G, \text{ one has } f(y) \in [f(x)]_G \text{ and } p(f(x), f(y)) \leq \alpha p(x, y).$$

Moreover, assume that one of the following conditions holds:

(i) if $f^n(x_0) \to x$ then $x = f(x)$;

(ii) if $f^n(x_0) \to x$ then there exists a subsequence $\left(f^{n_k}(x_0)\right)_{k\in\mathbb{N}}$ such that $p(f^{n_k}(x_0), x) \to 0$ and $f^{n_k}(x_0) \in [x]_G$ for $k \in \mathbb{N}$.

Then f has a fixed point.

Remark 1.3.2. *It is clear from the proof that Theorem 1.3.1 still holds if we replace the assumption that F is a weak G-contraction by the following condition:*
(wC_G)' for all $x \in X$ and $u \in F(x) \cap [x]_G$, there exists $v \in F(u) \cap [u]_G$ such that $p(u,v) \leq \alpha p(x,u)$.

Taking into account the previous remark, Theorem 1.3.1 generalizes the following result of Nicolae, O'Regan and Petruşel [34].

Corollary 1.3.2 (Nicolae, O'Regan and Petruşel [34])**.** *Let $F : X \to X$ be a multivalued map with nonempty closed values. Assume that*

(i) there exists $\lambda \in]0,1[$ such that

$$D(F(x), F(y)) \leq \lambda d(x,y) \quad \forall (x,y) \in E(G);$$

(ii) for each $x \in X$, if $u \in [x]_G \cap F(x)$, then for every $v \in F(u)$, there exists $(u^i)_{i=1}^N$, an N-directed path between $u = u^0$ and $v = u^N$, such that

$$d(u,v) = \sum_{i=1}^{N} d(u^{i-1}, u^i);$$

(iii) F has closed graph.

If there exist $x_0, x_1 \in X$ such that $x_1 \in [x_0]_G^1 \cap F(x_0)$, then F has a fixed point.

1.4. Comparison between the fixed points sets

We consider a multivalued weak G-contraction $F : X \to X$. In this section, we would like to compare the fixed point sets obtained by Picard iterations or by G-Picard trajectories from different points x_0 and y_0. To this aim, we introduce the following notation.

$$\mathrm{Fix}(F, x_0) = \{x : \exists (x_n)_{n\in\mathbb{N}} \text{ such that } x_n \to x \in F(x) \text{ and } x_n \in F(x_{n-1})\ \forall n \in \mathbb{N}\},$$
$$\mathrm{Fix}_G(F, x_0) = \{x : \exists (x_n)_{n\in\mathbb{N}} \in T(F, G, x_0) \text{ such that } x_n \to x \in F(x)\},$$
$$\mathrm{Fix}(F) = \{x : x \in F(x)\}.$$

Theorem 1.4.1. *Let $F : X \to X$ be a multivalued weak G-contraction and $x_0 \in X$. Then for every $y_0 \in [x_0]_G$,*

$$Fix(F, x_0) \subset Fix(F, y_0).$$

Proof. Let $x \in \mathrm{Fix}(F, x_0)$ and $(x_n)_{n\in\mathbb{N}}$ such that $x_n \to x$, $x \in F(x)$, and $x_n \in F(x_{n-1})$ for all $n \in \mathbb{N}$. Let $y_0 \in [x_0]_G$. So, there exists $(y_n)_{n\in\mathbb{N}}$ such that, for every $n \in \mathbb{N}$, $y_n \in F(y_{n-1})$, $y_n \in [x_n]_G$, and

$$d(x_n, y_n) \leq p(x_n, y_n) \leq \alpha^n p(x_0, y_0).$$

So, $y_n \to x \in F(x)$ and hence, $x \in \mathrm{Fix}(F, y_0)$. □

Theorem 1.4.2. *Let $F : X \to X$ be a multivalued weak G-contraction. Assume that $x_0 \in X$ is such that G_{x_0} is connected. Then for every $y_0 \in G_{x_0}$,*

$$Fix_G(F, x_0) = Fix_G(F, y_0).$$

In addition, if $[x_0]_G \cap F(x_0) \neq \emptyset$ and F is G-Picard continuous from x_0, then $Fix_G(F, x_0) \neq \emptyset$.

Proof. Let $x \in \mathrm{Fix}_G(F, x_0)$ and $(x_n)_{n\in\mathbb{N}} \in T(F, G, x_0)$ such that $x_n \to x$. Let $y_0 \in G_{x_0}$. By the proof of the previous theorem, we get $(y_n)_{n\in\mathbb{N}}$ such that $y_n \to x$, $y_n \in F(y_{n-1})$, and $y_n \in [x_n]_G$ for all $n \in \mathbb{N}$. Since G_{x_0} is connected, we deduce that $y_n \in [y_{n-1}]_G$ for all $n \in \mathbb{N}$. So $(y_n)_{n\in\mathbb{N}} \in T(F, G, y_0)$ and $x \in \mathrm{Fix}_G(F, y_0)$.

On the other hand, since G_{x_0} is connected, for $y_0 \in G_{x_0}$, one has $G_{x_0} = G_{y_0}$. Interchanging x_0 and y_0 in the previous argument permits us to deduce that $\mathrm{Fix}_G(F, x_0) = \mathrm{Fix}_G(F, y_0)$.

Finally, if $[x_0]_G \cap F(x_0) \neq \emptyset$ and F is G-Picard continuous from x_0, it follows from Theorem 1.3.1 that $\mathrm{Fix}_G(F, x_0) \neq \emptyset$. □

Corollary 1.4.1. *Let $F : X \to X$ be a multivalued weak G-contraction. Assume that G is connected. Then*

$$Fix(F) = Fix_G(F, y_0) \quad \forall y_0 \in X.$$

In addition, if F is G-Picard continuous from some x_0, then $Fix(F) \neq \emptyset$.

Remark 1.4.1. *We could have stated results analogous to the previous corollary and to Theorem 1.4.2, replacing the condition on the G-Picard continuity of F by assumption (ii) of Theorem 1.3.1.*

Let us point out that all the results in this section hold also for G-contractions.

1.5. Homotopical invariance and local results

In this section, we investigate the homotopical invariance of the fixed point property for a family of weak G-contractions. Let $I \subset [0, 1]$ be closed such that $\{0, 1\} \subset I$. Let $\hat{G}$ be a directed graph in $I \times X$ with no parallel edges such that $V(\hat{G}) = I \times X$ and, for every $t \in I$ and every $x, y \in X$,

$$\big((t, x), (t, y)\big) \in E(\hat{G}) \iff (x, y) \in E(G).$$

We first present a local fixed point result for weak G-contractions. We denote by $B(x, r)$ the open ball centered in x of radius r.

Proposition 1.5.1. *Let $F : B(x_0, r) \to X$ be a weak G-contraction with constant α. Assume that there exists $x_1 \in F(x_0) \cap [x_0]_G$ such that $p(x_0, x_1) < (1-\alpha)r$. In addition, assume that one of the following conditions is satisfied:*

(i) F is G-Picard continuous from x_0;

(ii) F has closed values and, for every $(x_n)_{n\in\mathbb{N}}$ in $T(F, G, x_0)$ such that $x_n \to x$, there exists $(n_k)_{k\in\mathbb{N}}$ such that $p(x_{n_k}, x) \to 0$ and $x \in [x_{n_k}]_G$ for $k \in \mathbb{N}$.

Then F has a fixed point x such that

$$d(x_0, x) \leq \frac{p(x_0, x_1)}{1-\alpha}.$$

Moreover, if (ii) is satisfied, $x \in [x_0]_G$.

Proof. Let $\hat{r} < r$ such that $p(x_0, x_1) < (1-\alpha)\hat{r}$. Since F is a weak G-contraction, there exists $x_2 \in [x_1]_G \cap F(x_1)$ such that

$$p(x_1, x_2) \leq \alpha\Big(p(x_0, x_1)\Big) < (1-\alpha)\hat{r}.$$

Moreover, $x_2 \in B(x_0, \hat{r})$. Indeed,

$$d(x_0, x_2) \leq d(x_0, x_1) + d(x_1, x_2) \leq p(x_0, x_1) + \alpha\Big(p(x_0, x_1)\Big) < (1-\alpha)\hat{r} + \alpha\hat{r} = \hat{r}.$$

Repeating the argument, we get a G-Picard trajectory, $(x_n)_{n\in\mathbb{N}}$ in $B(x_0, \hat{r})$ such that $p(x_n, x_{n+1}) \leq \alpha^n p(x_0, x_1)$ for every $n \in \mathbb{N}$. Thus, it is a Cauchy sequence which converges to some $x \in \overline{B(x_0, \hat{r})} \subset B(x_0, r)$ such that

$$d(x_0, x) \leq \frac{p(x_0, x_1)}{1-\alpha}.$$

Arguing as in the proof of Theorem 1.3.1, we deduce that x is a fixed point of F. Moreover, if (ii) is satisfied, $x \in [x_0]_G$. □

We obtain a homotopical invariance result for a family of weak G-contractions.

Theorem 1.5.1. *Let $\hat{x} \in X$, $r > 0$, and $H : I \times \overline{B(\hat{x}, r)} \to X$ a multivalued map with nonempty closed values such that*

(i) for all $t \in I$, $H(t, \cdot) : \overline{B(\hat{x}, r)} \to X$ is a weak G-contraction with constant α;

(ii) for every $t \in I$, $x_0 \in B(\hat{x}, r)$, and every $(x_n)_{n\in\mathbb{N}}$ in $T(H(t, \cdot), G, x_0)$ such that $x_n \to x$, there exists $(n_k)_{k\in\mathbb{N}}$ such that $p(x_{n_k}, x) \to 0$ and $x \in [x_{n_k}]_G$ for $k \in \mathbb{N}$;

(iii) there exists a continuous and nondecreasing map $g : I \to [0, \infty[$ such that for every $x \in B(\hat{x}, r)$, $s \in I \backslash \{1\}$ and every $u \in H(s, x)$, there exist $t > s$

and $v \in H(t,x)$ *such that* $(t,x) \in [(s,x)]_{\hat{G}}$, $v \in [u]_G$, *and*

$$p(u,v) \le |g(t) - g(s)| < \Big(1-\alpha\Big)\Big(r - d(x,\hat{x})\Big);$$

(iv) *for every* $(x_n)_{n\in\mathbb{N}}$ *in* X *and every nondecreasing sequence* $(t_n)_{n\in\mathbb{N}}$ *such that* $t_n \to t$, $x_n \to x$, $x_n \in H(t_n,x_n)$, *and* $(t_{n+1},x_{n+1}) \in [(t_n,x_n)]_{\hat{G}}$ *for every* $n \in \mathbb{N}$, *one has* $x \in H(t,x)$ *and* $(t,x) \in [(t_n,x_n)]_{\hat{G}}$ *for every* $n \in \mathbb{N}$;

(v) $x \notin H(t,x)$ *for all* $x \in \partial B(\hat{x},r)$ *and all* $t \in I$.

If $H(0,\cdot)$ *has a fixed point then* $H(1,\cdot)$ *has a fixed point.*

Proof. Let

$$Q = \{(t,x) \in I \times B(\hat{x},r) : x \in H(t,x)\}$$

be endowed with the partial order

$$(s,x) \le (t,y) \iff s \le t,\ (t,y) \in [(s,x)]_{\hat{G}},\ \text{and}\ d(x,y) \le \frac{|g(t)-g(s)|}{1-\alpha}.$$

Let P be a chain in Q. We claim that P has an upper bound. Indeed, let $t^* = \sup\{t : (t,x) \in P\}$. There exists $(t_n,x_n) \in P$ such that $(t_n,x_n) \le (t_{n+1},x_{n+1})$ and $t_n \to t^*$. So,

$$d(x_n,x_{n+m}) \le \frac{|g(t_{n+m}) - g(t_n)|}{1-\alpha} \quad \forall m,n \in \mathbb{N}.$$

The continuity of g implies that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence which converges to some x^*. By (iv) and (v),

$$x^* \in H(t^*,x^*) \cap B(\hat{x},r), < (t^*,x^*) \in Q, \quad \text{and} \quad (t,x) \le (t^*,x^*) \quad \forall (t,x) \in P.$$

Since $H(0,\cdot)$ has a fixed point, $Q \neq \emptyset$. By Zorn's Lemma, Q has a maximal element (s,x_0).

To conclude, we have to show that $s = 1$. Assume $s < 1$. By (v), $x_0 \in B(\hat{x},r)$. Assumption (iii) implies that there exist $t > s$ and $x_1 \in H(t,x_0)$ such that $x_1 \in [x_0]_G$, $(t,x_0) \in [(s,x_0)]_{\hat{G}}$, and

$$p(x_0,x_1) \le |g(t) - g(s)| < \Big(1-\alpha\Big)\Big(r - d(\hat{x},x_0)\Big).$$

Let $\delta = r - d(\hat{x},x_0)$. One has that x_1 and $H(t,\cdot) : B(x_0,\delta) \to X$ satisfy the assumptions of Proposition 1.5.1. So there exists $x \in H(t,x)$ such that $x \in [x_0]_G$ and

$$d(x,x_0) \le \frac{p(x_0,x_1)}{1-\alpha} \le \frac{|g(t)-g(s)|}{1-\alpha}.$$

Therefore $(t,x) \in Q$, $(t,x) \in [(s,x_0)]_{\hat{G}}$ and hence, $(s,x_0) \le (t,x)$. This contradicts the maximality of (s,x_0). □

We obtain the following corollary for single-valued maps.

Corollary 1.5.1. *Let $\hat{x} \in X$, $r > 0$, and $h : I \times \overline{B(\hat{x}, r)} \to X$ a single-valued map such that*

(i) there exists $\alpha \in \,]0, 1[$ such that for all $x, y \in B(\hat{x}, r)$ with $y \in [x]_G$, one has $p\Big(h(t, x), h(t, y)\Big) \leq \alpha p(x, y)$ for all $t \in I$;

(ii) for every $t \in I$ and $x_0 \in B(\hat{x}, r)$, if $\Big(h_t^n(x_0)\Big)_{n\in\mathbb{N}} \to x$, there exists $(n_k)_{k\in\mathbb{N}}$ such that $p\Big(h_t^{n_k}(x_0), x\Big) \to 0$ and $x \in [h_t^{n_k}(x_0)]_G$ for $k \in \mathbb{N}$, where $h_t(\cdot) = h(t, \cdot)$;

(iii) there exists a continuous and nondecreasing map $g : I \to [0, \infty[$ such that for every $x \in B(\hat{x}, r)$, $s \in I \backslash \{1\}$, there exists $t > s$ such that $(t, x) \in [(s, x)]_{\hat{G}}$, $h(t, x) \in [h(s, x)]_G$, and

$$p\Big(h(t, x), h(s, x)\Big) \leq |g(t) - g(s)| < \Big(1 - \alpha\Big)\Big(r - d(x, \hat{x})\Big);$$

(iv) for every $(x_n)_{n\in\mathbb{N}}$ in X and every nondecreasing sequence $(t_n)_{n\in\mathbb{N}}$ such that $t_n \to t$, $x_n = h(t_n, x_n) \to x$, and $(t_{n+1}, x_{n+1}) \in [(t_n, x_n)]_{\hat{G}}$ for every $n \in \mathbb{N}$, one has $x = h(t, x)$ and $(t, x) \in [(t_n, x_n)]_{\hat{G}}$ for every $n \in \mathbb{N}$;

(v) $x \neq h(t, x)$ for all $x \in \partial B(\hat{x}, r)$ and all $t \in I$.

If $h(0, \cdot)$ has a fixed point then $h(1, \cdot)$ has a fixed point.

1.6. Strong comparison functions

It is well known that many fixed point results were obtained for multivalued maps satisfying some generalization of the classical contraction condition. One of them is due to Węgrzyk [47] and relies on strong comparison functions.

Definition 1.6.1. *A map $\varphi : [0, \infty[\, \to [0, \infty[$ is a strong comparison function if φ is increasing, $\varphi(0) = 0$, and*

$$\sum_{n=0}^{\infty} \varphi^n(t) \text{ converges} \quad \forall t > 0.$$

Using such a function, we can introduce the notion of (G, φ)-contraction.

Definition 1.6.2. *Let $\varphi : [0, \infty[\, \to [0, \infty[$ be a strong comparison function. We say that a multivalued map with nonempty values, $F : X \to X$, is a (G, φ)-contraction if*

(φC_G) for all $(x, y) \in E(G)$ and all $u \in F(x)$, there exists $v \in F(y)$ such that $(u, v) \in E(G)$ and $d(u, v) \leq \varphi\Big(d(x, y)\Big)$.

Here is a generalization of Theorem 1.2.1 and Corollary 1.2.1.

Theorem 1.6.1. *Let $\varphi : [0, \infty[\, \to [0, \infty[$ be a strong comparison function and let $F : X \to X$ be a multivalued (G, φ)-contraction. Assume that there exist $N \in \mathbb{N}$ and x_0 such that $[x_0]_G^N \cap F(x_0) \neq \emptyset$. Assume also that one of the following assumptions is satisfied:*

(i) F is G_N-Picard continuous from x_0;

(ii) F has closed values and, for every $(x_n)_{n\in\mathbb{N}}$ in $T_N(F,G,x_0)$ such that $x_n \to x$, there exists $(n_k)_{k\in\mathbb{N}}$ such that $(x_{n_k}, x) \in E(G)$ for $k \in \mathbb{N}$.

Then there exists a G_N-Picard trajectory $(x_n)_{n\in\mathbb{N}}$ converging to x a fixed point of F.

Proof. Let $x_1 \in [x_0]_G^N \cap F(x_0)$ and $(x^i)_{i=0}^N$ an N-directed path from $x_0 = x^0$ to $x_1 = x^N$. Since F is a (G,φ)-contraction and arguing as in Lemma 1.2.1, we get $(x_n)_{n\in\mathbb{N}} \in T_N(F,G,x_0)$ such that

$$d(x_n, x_{n+1}) \le p_N(x_n, x_{n+1}) \le \sum_{i=1}^{N} \varphi^n\Big(d(x^{i-1}, x^i)\Big) \quad \forall n \in \mathbb{N}.$$

We deduce from the fact that φ is a strong comparison function that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence which converges to some $x \in X$.

If (i) is satisfied, x is a fixed point of F.

On the other hand, if (ii) is satisfied, since F is a (G,φ)-contraction, there exist $(n_k)_{k\in\mathbb{N}}$ and $(y_{n_k+1})_{k\in\mathbb{N}}$ such that

$$(x_{n_k}, x) \in E(G),\ (x_{n_k+1}, y_{n_k+1}) \in E(G),\ y_{n_k+1} \in F(x),$$

and

$$d(x_{n_k+1}, y_{n_k+1}) \le \varphi\Big(d(x_{n_k}, x)\Big) \quad \forall k \in \mathbb{N}.$$

This inequality with the fact that φ is a strong comparison function imply that $d(x_{n_k+1}, y_{n_k+1}) \to 0$ and hence, $y_{n_k+1} \to x$. Finally, $x \in F(x)$ since F has closed values. $\square$

Also, a notion of weak (G,φ)-contraction can be introduced similarly to the notions introduced above.

Definition 1.6.3. *Let $\varphi : [0,\infty[\to [0,\infty[$ be a strong comparison function. We say that a multivalued map with nonempty values, $F : X \to X$, is a weak (G,φ)-contraction if for all $x \in X$, $y \in [x]_G$, and all $u \in F(x)$, there exists $v \in F(y)$ such that $v \in [u]_G$ and $p(u,v) \le \varphi\Big(p(x,y)\Big)$.*

Arguing as in the proofs of Theorems 1.3.1 and 1.6.1, we obtain the following result.

Theorem 1.6.2. *Let $\varphi : [0,\infty[\to [0,\infty[$ be a strong comparison function and $F : X \to X$ a multivalued weak (G,φ)-contraction. Assume there exists $x_0 \in X$ such that $[x_0]_G \cap F(x_0) \ne \emptyset$. In addition, assume that one of the following condition holds:*

(i) F is G-Picard continuous from x_0;

(ii) *F has closed values and, for every $(x_n)_{n\in\mathbb{N}}$ in $T(F,G,x_0)$ such that $x_n \to x$, there exists $(n_k)_{k\in\mathbb{N}}$ such that $p(x_{n_k},x) \to 0$ and $x \in [x_{n_k}]_G$ for $k \in \mathbb{N}$.*

Then there exists a G-Picard trajectory $(x_n)_{n\in\mathbb{N}}$ converging to x a fixed point of F.

Chapter 2

SYSTEMS OF HAMMERSTEIN INTEGRAL INCLUSIONS IN BANACH SPACES WITH MIXED MONOTONE CONDITIONS

2.1. Introduction

We consider the following system of Hammerstein integral inclusions:

$$x_i(t) \in \int_0^1 H_i(t, s, x_1(s), \ldots, x_N(s))\, ds \quad \forall t \in [0,1],\ \forall i = 1, \ldots, N. \tag{2.1.1}$$

Here, $H_i : [0,1] \times [0,1] \times E_1 \times \cdots \times E_N \to E_i$ are multivalued maps with nonempty values and E_i are Banach spaces.

Existence results for this problem have been established by many authors when $N = 1$. In particular, in [6], [38], [39], the existence of a solution was obtained by applying a set-valued Mönch type fixed point theorem for multivalued maps while it was deduced from a fixed point theorem for condensing multivalued maps in [3] and [41]. Lispschitz type conditions were imposed on H_1 in [3] and [42], where existence results were deduced from Nadler fixed point results for multivalued contractions and for (ε, λ)-uniformly locally contractive multivalued maps respectively. In [27], Hong and Qui considered the case where E_1 is endowed with a partial order. Assuming that H_1 is integrably bounded, increasing with respect to x_1 and assuming the existence of lower and upper solutions, he deduced the existence of a solution from a Mönch type result for multivalued maps in ordered Banach spaces.

In 2004, Ran and Reurings [44] established a fixed point result which is, in some sense, a combination of the Banach contraction principle and the Knaster-Tarski fixed point theorem in a partially ordered set. They considered a continuous, monotone, order preserving single-valued map f defined on a complete metric space endowed with a partial ordering. They assumed that f satisfies a contraction condition not necessarily for all x and y, but for those such that $x \leq y$. Their result was generalized by Nieto and Rodríguez-López [36], [37] who weakened the continuity and the monotonicity assumptions. Later, Jachymski [29] presented an unification of the previous results by considering complete metric spaces endowed with a graph G. He introduced the notion of single-valued G-contraction for which he obtained fixed point results. Those results were extended to multivalued maps on complete metric space endowed with a graph in [10]. In particular, the notion of multivalued weak G-contraction was introduced. This notion was new even in the single-valued case.

Nieto and Rodríguez-López [36], [37] applied their fixed point results to periodic boundary value problems for a first order differential equation with a monotone right hand side satisfying a Lipschitz type condition. Assuming the existence of a lower-solution (or an upper-solution), they established the existence of a solution.

On the other hand, Guo and Lakshmikantham [24] were the first to obtain the existence of a coupled fixed point to a mixed monotone single-valued operator $T : D \times D \to E_1$ for $D \subset E_1$ and E_1 endowed with a partial order. Inspired by the results in [24], [36], [37] and [44], Gnana Bhaskar and Lakshimikantham [23] established a coupled fixed point result for a mixed monotone operator satisfying a contraction-type condition. They applied their result to a periodic boundary value problem for a first order differential equation under a mixed monotone condition and under the existence of a coupled lower and upper solutions. In [26], fixed point results were obtained for multivalued mixed monotone operators. Those results were applied to establish the existence of a solution to an initial value problem for a system of differential inclusions with the right hand side satisfying a mixed monotone condition.

In this paper, we study the problem (2.1.1) with Banach spaces E_i endowed with a partial order. We do not assume that the maps H_i have closed or compact values. Our main existence result is established in Section 3 where we assume that the multivalued maps H_i are nondecreasing or nonincreasing with respect to each variable x_j. The continuity type condition imposed on H_i is weaker than the notion upper semi-continuity. In Sections 4 and 5, we consider the particular cases where the maps H_i are respectively nonincreasing and nondecreasing (upward or downward). This permits us to obtain existence results with a weaker continuity type condition. This condition can be made even weaker if the order on E_i satisfies an extra condition. In Section 6, we consider the case where some maps H_i can be nondecreasing and nonincreasing with respect to some x_j. Our existence results rely on a slight modification of the fixed point result for multivalued weak G-contractions obtained in [10] and which is presented in Section 2. The application of this fixed point result to our problem is very natural and appropriate. It is worth to point out that we do not use the theory of coupled fixed point results for mixed monotone operators.

In what follows, for E a Banach space, the space of continuous functions from $[0,1]$ to E is denoted by $C([0,1],E)$ and endowed with the usual norm $\|u\|_0 = \max\{\|u(t)\| : t \in [0,1]\}$. For $p \in [0,\infty[$, we consider $L^p([0,1],E)$ the space of Bochner measurable functions $u : [0,1] \to E$ such that $\|u\|^p$ is Lebesgue integrable on $[0,1]$, and we denote $\|u\|_p = \int_0^1 \|u(s)\|^p\, ds$. For $p=\infty$, $L^\infty([0,1],E)$ is the space of Bochner measurable functions which are essentially bounded. This space is endowed with the usual norm $\|u\|_\infty$. We denote by $W^{1,1}([0,1],E)$ the Sobolev space of absolutely continuous functions $u : [0,1] \to E$ such that $u' \in L^1([0,1],E)$.

2.2. Multivalued contractions on a metric space endowed with a graph

We recall some notions and results concerning multivalued contractions on a metric space endowed with a graph obtained in [10].

2.2.1. Definitions and notations

Let (X,d) be a complete metric space. We consider a directed graph G such that the set of its vertices $V(G) = X$ and the set of its edges $E(G)$ has no parallel edges and contains Δ the diagonal in $X \times X$. We identify G with the pair $(V(G),E(G))$.

For $x,y \in X$ and $m \in \mathbb{N}$, $(x^i)_{i=0}^m$ is called an *m-directed path* from x to y if $x = x^0$, $y = x^m$, and $(x^{i-1},x^i) \in E(G)$ for every $i = 1,\dots,m$. We denote

$$[x]_G^m = \{y \in X : \text{ there is an } m\text{-directed path from } x \text{ to } y\},$$
$$[x]_G = \bigcup_{m\in\mathbb{N}} [x]_G^m.$$

Observe that $[x]_G^1 \subset [x]_G^2 \subset \cdots \subset [x]_G$ since $\Delta \subset E(G)$.

For $y \in [x]_G^m$ and $z \in [x]_G$, we define

$$p_m(x,y) = \inf\left\{ \sum_{i=1}^m d(x^{i-1},x^i) : (x^i)_{i=0}^m \text{ is an } m\text{-directed path from } x \text{ to } y \right\};$$

$$p(x,z) = \inf\left\{ \sum_{i=1}^m d(x^{i-1},x^i) : (x^i)_{i=0}^m \text{ is an } m\text{-directed path from } x \text{ to } z \text{ for some } m \in \mathbb{N} \right\}.$$

Notice that $p_m(x,y) \geq p_{k+m}(x,y)$ for all $k \in \mathbb{N}$, and

$$p(x,z) = \inf\{p_m(x,z) : m \in \mathbb{N} \text{ such that } z \in [x]_G^m\}$$

since $\Delta \subset E(G)$.

Definition 2.2.1. *Let $F : X \to X$ be a multivalued mapping.*

(1) Let $m \in \mathbb{N}$. We say that a sequence $\{x_n\}$ is a G_m-Picard trajectory from x_0 if $x_n \in [x_{n-1}]_G^m \cap F(x_{n-1})$ for all $n \in \mathbb{N}$. We denote by $T_m(F, G, x_0)$, the set of all G_m-Picard trajectories from x_0.

(2) We say that a sequence $\{x_n\}$ is a G-Picard trajectory from x_0 if $x_n \in [x_{n-1}]_G \cap F(x_{n-1})$ for all $n \in \mathbb{N}$. We denote by $T(F, G, x_0)$, the set of all G-Picard trajectories from x_0.

Definition 2.2.2. *Let $F : X \to X$ be a multivalued mapping.*

(1) Let $m \in \mathbb{N}$. We say that F is G_m-Picard continuous from x_0 if the limit of any convergent sequence $\{x_n\} \in T_m(F, G, x_0)$ is a fixed point of F.

(2) We say that F is G-Picard continuous from x_0 if the limit of any convergent sequence $\{x_n\} \in T(F, G, x_0)$ is a fixed point of F.

We recall the notions of contractions with respect to G introduced in [10].

Definition 2.2.3. *Let $Y \subset X$ and $F : Y \to X$ a multivalued mapping with nonempty values.*

(1) We say that F is a G-contraction if there exists $\lambda \in\,]0,1[$ such that, for all $(x,y) \in E(G)$ and all $u \in F(x)$, there exists $v \in F(y)$ such that

$$(u,v) \in E(G) \quad \textit{and} \quad d(u,v) \leq \lambda d(x,y).$$

(2) We say that F is a weak G-contraction if there exists $\lambda \in\,]0,1[$ such that, for all $x, y \in Y$ with $y \in [x]_G$, and all $u \in F(x)$, there exists $v \in F(y)$ such that

$$v \in [u]_G \quad \textit{and} \quad p(u,v) \leq \lambda p(x,y).$$

It is easy to verify that a G-contraction is a weak G-contraction. An example of a weak G-contraction which is not a G-contraction is presented in [10].

2.2.2. Fixed point results for G-contractions

Here is a fixed point result for G-contraction established in [10].

Theorem 2.2.1. *Let $F : X \to X$ be a multivalued G-contraction. Assume there exist $m \in \mathbb{N}$ and $x_0 \in X$ such that $[x_0]_G^m \cap F(x_0) \neq \emptyset$ and F is G_m-Picard continuous from x_0. Then there exists a G_m-Picard trajectory $\{x_n\}$ converging to x a fixed point of F.*

Remark 2.2.1. *(1) Arguing as in [10], it can be shown that if $F : X \to X$ is a multivalued G-contraction, then there exists $\alpha \in \,]0,1[$ such that, for all $y \in [x]_G^m$ and all $u \in F(x)$, there exists $v \in F(y) \cap [u]_G^m$ such that $p_m(u,v) \leq \alpha p_m(x,y)$.*

(2) The assumption of G_m-Picard continuity concerns

$$\{\{x_n\} \in T_m(F,G,x_0) : \{x_n\} \text{ converges}\}.$$

Looking at the proof of Theorem 2.2.1, one sees that it is sufficient to have that the limit of a sequence in the following set is a fixed point of F:

$$\Big\{\{x_n\} \in T_m(F,G,x_0) : \sum_{n=1}^{\infty} p_m(x_{n-1},x_n) < \infty\Big\}.$$

So, the assumption of G_m-Picard continuity can be weaken in Theorem 2.2.1.

In practice, one realizes that it happens that the pair (u,v) in the definition of G-contraction satisfies also some other properties. Taking into account this fact and the previous remark, one can state a generalization of Theorem 2.2.1. Its proof is analogous to the proof of Theorem 2.2.1 and it is left to the reader.

Theorem 2.2.2. *Let $F : X \to X$ be a multivalued map and, for every $(x,y) \in E(G)$, let a property $\mathcal{P}(x,y)$. Assume there exists $m \in \mathbb{N}$ such that the following conditions hold:*

(i) there exist $x_0, x_1 \in X$ such that $x_1 \in [x_0]_G^m \cap F(x_0)$;

(ii) there exists $\alpha \in \,]0,1[$ such that, for all $y \in [x]_G^m$ and all $u \in F(x)$, there exists $v \in F(y) \cap [u]_G^m$ such that

$$p_m(u,v) \leq \alpha p_m(x,y), \quad \text{and} \quad (u,v) \text{ satisfies the property } \mathcal{P}(x,y).$$

(iii) for any sequence $\{x_n\} \in T_m(F,G,x_0)$ (with x_0, x_1 given in (i)) such that (x_{n+1},x_{n+2}) satisfies property $\mathcal{P}(x_n,x_{n+1})$ for every $n \in \mathbb{N}$ and

$$\sum_{n=1}^{\infty} p_m(x_{n-1},x_n) < \infty,$$

its limit is a fixed point of F.

Then there exists a G_m-Picard trajectory $\{x_n\}$ converging to x a fixed point of F.

2.2.3. Fixed point results for weak G-contractions

Here is a fixed point result for weak G-contractions obtained in [10].

Theorem 2.2.3. *Let $F : X \to X$ be a multivalued weak G-contraction. Assume there exists $x_0 \in X$ such that $[x_0]_G \cap F(x_0) \neq \emptyset$ and F is G-Picard continuous from x_0. Then there exists a G-Picard trajectory $\{x_n\}$ converging to x a fixed point of F.*

It is important to understand that if F is a G-contraction then for $(x, y) \in E(G)$ and $u \in F(x)$, there is $v \in F(y)$ such that there is a 1-directed path from u to v such that $d(u, v) \leq \lambda d(x, y)$. On the other hand, if F is a weak G-contraction, for $(x, y) \in E(G)$ and $u \in F(x)$, one cannot insure that there is an appropriate element of $F(y)$ on a 1-directed path from u. Indeed, a suitable $v \in F(y)$ could be on an N-directed path from u for some N strictly bigger than 1. A particular case of weak G-contraction is when such N is the same for all $(x, y) \in E(G)$. Also, it could happen in practice that (u, v) satisfies some other properties. The proof of the following result is analogous to the proof of Theorem 2.2.3 (see the proof of Theorem 3.4 in [10]) and it is left to the reader.

Theorem 2.2.4. *Let $F : X \to X$ be a multivalued map and $m, N \in \mathbb{N}$. Let $\mathcal{P}_k(x, y)$ be a property for every $y \in [x]_G^k$ for $k = mN^n$ with $n \in \mathbb{N} \cup \{0\}$. Assume the following assumptions:*

(i) There exists $x_0 \in X$ such that $[x_0]_G^m \cap F(x_0) \neq \emptyset$.

(ii) There exists $\alpha \in]0, 1[$ such that for every $x \in X$, $u \in F(x)$ and every $y \in [x]_G^k$ with $k = mN^n$ and $n \in \mathbb{N} \cup \{0\}$, there exists $v \in [u]_G^{kN} \cap F(y)$ such that

$$p_{kN}(u, v) \leq \alpha p_k(x, y) \quad \textit{and} \quad (u, v) \textit{ satisfies the property } \mathcal{P}_k(x, y).$$

(iii) For any sequence $\{x_n\}$ (with x_0 given in (i)) such that $x_n \in F(x_{n-1}) \cap [x_{n-1}]_G^{mN^{n-1}}$, (x_n, x_{n+1}) satisfies property $\mathcal{P}_{mN^{n-1}}(x_{n-1}, x_n)$ for every $n \in \mathbb{N}$, and

$$\sum_{n=1}^{\infty} p_{mN^n}(x_n, x_{n+1}) < \infty,$$

its limit is a fixed point of F.

Then there exists a G-Picard trajectory $\{x_n\}$ converging to x a fixed point of F.

2.3. Systems of Hammerstein integral inclusions with mixed monotonicity type conditions

We consider the system of Hammerstein integral inclusions (2.1.1):

$$x_i(t) \in \int_0^1 H_i(t, s, x_1(s), \ldots, x_N(s))\, ds \quad \forall t \in [0, 1],\ \forall i = 1, \ldots, N,$$

where $H_i : [0, 1] \times [0, 1] \times E_1 \times \cdots \times E_N \to E_i$ are multivalued maps with nonempty values.

We assume that the Banach spaces E_i are endowed with a partial order $\preceq$ satisfying:

(O) for $u, v \in L^1([0,1], E_i)$ such $u(s) \preceq v(s)$ a.e. $s \in [0,1]$, one has

$$\int_0^1 u(s)\,ds \preceq \int_0^1 v(s)\,ds.$$

In this section, we establish existence results in the case where the maps H_i satisfy monotonicity type conditions with respect to each variable x_j. This could be nondecreasing type conditions with respect to some variables and nonincreasing type conditions with respect to the others.

We denote $E = E_1 \times \cdots \times E_N$ the Banach space endowed with the norm $\|(x_1, \ldots, x_N)\| = \|x_1\| + \cdots + \|x_N\|$, and $H : [0,1] \times [0,1] \times E \to E$ the multivalued map defined by $H(t,s,x) = (H_1(t,s,x), \ldots, H_N(t,s,x))$.

We define the multivalued map $\mathcal{H} : C([0,1], E) \to \mathcal{E}$ by

$$\mathcal{H}(x) = \Big\{ w \in \mathcal{E} : w(t,s) \in H(t,s,x(s)) \text{ a.e. } s \in [0,1],\ \forall t \in [0,1] \Big\}, \tag{2.3.1}$$

where

$$\mathcal{E} = \Big\{ w : [0,1] \times [0,1] \to E : w(t,\cdot) \in L^1([0,1], E)\ \forall t \in [0,1], \\ \text{and } t \mapsto \int_0^1 w(t,s)\,ds \text{ is continuous} \Big\}. \tag{2.3.2}$$

We look for solutions of (2.1.1) which are fixed points of the multivalued map $F : C([0,1], E) \to C([0,1], E)$ defined by

$$F(x) = \Big\{ u \in C([0,1], E) : \text{ there exists } w \in \mathcal{H}(x) \\ \text{such that } u(t) = \int_0^1 w(t,s)\,ds \Big\}. \tag{2.3.3}$$

Here is our first existence result.

Theorem 2.3.1. *Let $H : [0,1] \times [0,1] \times E \to E$ be a multivalued map with nonempty values. Assume the following conditions hold:*

(i) There exist $x_0 \in C([0,1], E)$, $w_0 \in \mathcal{H}(x_0)$ and $\sigma_0 : \{1, \ldots, N\} \to \{1, -1\}$ such that

$$0 \preceq \sigma_0(i) \left(\int_0^1 w_{0,i}(t,s)\,ds - x_{0,i}(t) \right) \quad \forall t \in [0,1],\ \forall i = 1, \ldots, N.$$

(ii) There exists $\phi : [0,1] \times [0,1] \to [0,\infty[$ such that

$$\phi(t,\cdot) \in L^1([0,1])\ \forall t \in [0,1] \quad \text{and} \quad \sup_{t \in [0,1]} \|\phi(t,\cdot)\|_1 < 1,$$

and, for $j = 1, \ldots, N$, there is a map $\sigma_j : \{1, \ldots, N\} \to \{1, -1\}$ such that, for every $x = (x_1, \ldots, x_N) \in C([0,1], E)$, $w \in \mathcal{H}(x)$ and every $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_N) \in C([0,1], E)$ such that $x_i = \hat{x}_i$ for $i \neq j$ and $x_j(s) \preceq \hat{x}_j(s)$

for all $s \in [0,1]$ *(resp.* $\hat{x}_j(s) \preceq x_j(s)$ *for all* $s \in [0,1]$*), there exists* $\hat{w} \in \mathcal{H}(\hat{x})$ *such that, for a.e.* $s \in [0,1]$ *and all* $t \in [0,1]$*,*

$$\|w(t,s) - \hat{w}(t,s)\| \le \phi(t,s)\|x - \hat{x}\|_0,$$

and

$$0 \preceq \sigma_j(i)(\hat{w}_i(t,s) - w_i(t,s))$$
$$\text{(resp. } 0 \preceq \sigma_j(i)(w_i(t,s) - \hat{w}_i(t,s))) \quad \forall i = 1, \dots, N).$$

(iii) For every $x, x_n \in C([0,1], E)$*,* $w \in \mathcal{E}$ *and* $w_n \in \mathcal{H}(x_n)$*, one has* $w \in \mathcal{H}(x)$ *if*

(a) $x_n \to x$ *and* $x_n(t) = \int_0^1 w_{n-1}(t,s)\,ds$ *for all* $t \in [0,1]$ *and all* $n \in \mathbb{N}$*;*

(b) $w_n(t,s) \to w(t,s)$ *and* $\|w_n(t,s)\| \le M\phi(t,s) + \|w_0(t,s)\|$ *a.e.* $s \in [0,1]$*, all* $t \in [0,1]$*, all* $n \in \mathbb{N}$*, and for some* $M \ge 0$*.*

Then, (2.1.1) *has a solution.*

Proof. We consider on $C([0,1], E)$ the following graph G with $V(G) = C([0,1], E)$ and

(E(G)) one has $\big((x_1, \dots, x_N), (y_1, \dots, y_N)\big) \in E(G)$ if and only if one of the following conditions holds:

(i) there exists $j \in \{1, \dots, N\}$ such that, for all $s \in [0,1]$, $x_j(s) \preceq y_j(s)$ and $x_i(s) = y_i(s)$ for all $i \neq j$;

(ii) there exists $j \in \{1, \dots, N\}$ such that, for all $s \in [0,1]$, $y_j(s) \preceq x_j(s)$ and $x_i(s) = y_i(s)$ for all $i \neq j$.

Let

$$x_1(t) = \int_0^1 w_0(t,s)\,ds \quad \forall t \in [0,1],$$

where w_0 is given in Assumption (i). One has $x_1 \in F(x_0)$. Observe that

$$(x_{0,1}, x_{0,2}, \dots, x_{0,N}), (x_{1,1}, x_{0,2}, \dots, x_{0,N}), \dots, \dots, (x_{1,1}, \dots, x_{1,N})$$

is an N-directed path from x_0 to x_1. So, $x_1 \in F(x_0) \cap [x_0]_G^N$.

We consider the following properties:

(P_1) For $(x, \hat{x}) \in E(G)$, we say that $(u, \hat{u}) \in \mathcal{P}_1(x, \hat{x})$ if, for all $w \in \mathcal{H}(x)$ such that $u(t) = \int_0^1 w(t,s)\,ds$ for all $t \in [0,1]$, there exists $\hat{w} \in \mathcal{H}(\hat{x})$ such that

(a) $\hat{u}(t) = \int_0^1 \hat{w}(t,s)\,ds$ for all $t \in [0,1]$;

(b) $\|w(t,s) - \hat{w}(t,s)\| \le \phi(t,s)\|x - \hat{x}\|_0$ a.e. $s \in [0,1]$ and all $t \in [0,1]$.

(P_N) For $y \in [x]_G^N$, we say that $(u, v) \in \mathcal{P}_N(x, y)$ if there exist $(x^k)_{k=0}^N$ and $(u^k)_{k=0}^N$ N-directed paths from x to y and from u to v respectively such that $(u^{k-1}, u^k) \in \mathcal{P}_1(x^{k-1}, x^k)$ for $k = 1, \dots, N$. Hence, for all $w^0 \in \mathcal{H}(x)$

such that $u(t) = \int_0^1 w^0(t,s)\,ds$, there exists $w^k \in \mathcal{H}(x^k)$ such that

$$u^k(t) = \int_0^1 w^k(t,s)\,ds$$

and

$$\|w^0(t,s) - w^N(t,s)\| \le \sum_{k=1}^{N} \|w^{k-1}(t,s) - w^k(t,s)\| \le \phi(t,s)p_N(x,y)$$
$$\text{a.e. } s \in [0,1], \forall t \in [0,1].$$

(P_{N^n}) For $n \ge 2$ and $y \in [x]_G^{N^n}$, the property $\mathcal{P}_{N^n}(x,y)$ is defined inductively. Hence, for $(u,v) \in \mathcal{P}_{N^n}(x,y)$, one has $v \in [u]_G^{N^{n+1}}$ and for all $w \in \mathcal{H}(x)$ such that $u(t) = \int_0^1 w(t,s)\,ds$, there exists $\hat{w} \in \mathcal{H}(y)$ such that $v(t) = \int_0^1 \hat{w}(t,s)\,ds$ and

$$\|w(t,s) - \hat{w}(t,s)\| \le \phi(t,s)p_{N^n}(x,y) \quad \text{a.e. } s \in [0,1],\ \forall t \in [0,1].$$

We claim that F satisfies Condition (ii) of Theorem 2.2.4 with

$$\alpha \in \left]\sup_{t\in[0,1]} \|\phi(t,\cdot)\|_1, 1\right[.$$

Indeed, let $(x,\hat{x}) \in E(G)$ and $u \in F(x)$. For $w \in \mathcal{H}(x)$ such that

$$u(t) = \int_0^1 w(t,s)\,ds \quad \forall t \in [0,1],$$

let $\hat{w} \in \mathcal{H}(\hat{x})$ be insured by Assumption (ii) and define

$$\hat{u}(t) = \int_0^1 \hat{w}(t,s)\,ds \quad \forall t \in [0,1].$$

Thus, by (O), for $i = 1,\dots,N$,

$$\begin{aligned} u_i(t) &\preceq \hat{u}_i(t) \quad \forall t \in [0,1], \\ \text{or} \quad \hat{u}_i(t) &\preceq u_i(t) \quad \forall t \in [0,1]. \end{aligned}$$

One has $\hat{u} \in [u]_G^N$ since

$$(u_1,\dots,u_N), (\hat{u}_1, u_2, \dots, u_N), \dots, (\hat{u}_1, \dots, \hat{u}_N)$$

is an N-directed path from u to $\hat{u}$. Also, $(u,\hat{u}) \in \mathcal{P}_1(x,\hat{x})$ and

$$\begin{aligned} p_N\Big(u,\hat{u}\Big) &\le \Big\|(u_1,\dots,u_N) - (\hat{u}_1, u_2, \dots, u_N)\Big\|_0 + \cdots \\ &\qquad + \Big\|(\hat{u}_1,\dots,\hat{u}_{N-1}, u_N) - (\hat{u}_1,\dots,\hat{u}_N)\Big\|_0 \\ &= \|u - \hat{u}\|_0 \\ &\le \lambda\|x - \hat{x}\|_0, \end{aligned}$$

with

$$\lambda = \sup_{t\in[0,1]} \|\phi(t,\cdot)\|_1.$$

Fix $\varepsilon > 0$ such that $\lambda(1+\varepsilon) < \alpha$. Let $y \in [x]_G^N$ and $u \in F(x)$. There exists $(x^k)_{k=0}^N$ an N-directed path from x to y such that

$$\sum_{k=1}^{N} \|x^{k-1} - x^k\|_0 \le (1+\varepsilon) p_N(x,y).$$

The previous argument insures that, for $k = 0, \ldots, N$, there exists $w^k \in \mathcal{H}(x^k)$ such that $u = u^0$,

$$u^k(t) = \int_0^1 w^k(t,s)\, ds \quad \forall t \in [0,1],$$

and $(u^{k-1}, u^k) \in \mathcal{P}_1(x^{k-1}, x^k)$. So, for $v = u^N$, one has $(u,v) \in \mathcal{P}_N(x,y)$, $v \in F(y) \cap [u]_G^{N^2}$ and

$$\begin{aligned} p_{N^2}(u,v) &\le \sum_{k=1}^{N} p_N(u^{k-1}, u^k) \\ &\le \lambda \sum_{k=1}^{N} \|x^{k-1} - x^k\| \\ &\le \alpha p_N(x,y). \end{aligned}$$

By induction on n, it can be shown that for every $y \in [x]_G^{N^n}$ and every $u \in F(x)$, there exists $v \in F(y) \cap [u]_G^{N^{n+1}}$ such that $(u,v) \in \mathcal{P}_{N^n}(x,y)$ and

$$p_{N^{n+1}}(u,v) \le \alpha p_{N^n}(x,y).$$

Finally, (iii) implies that Condition (iii) of Theorem 2.2.4 is satisfied. Indeed, let $\{x_n\}$ be such that $x_n \in F(x_{n-1}) \cap [x_{n-1}]_G^{N^n}$ and $(x_n, x_{n+1}) \in \mathcal{P}_{N^n}(x_{n-1}, x_n)$ for all $n \in \mathbb{N}$ and

$$\sum_{n=1}^{\infty} p_{N^{n+1}}(x_n, x_{n+1}) < \infty.$$

Since

$$\sum_{n=1}^{\infty} \|x_n - x_{n+1}\|_0 \le \sum_{n=1}^{\infty} p_{N^{n+1}}(x_n, x_{n+1}),$$

$\{x_n\}$ is a Cauchy sequence converging to some $x \in C([0,1], E)$. Also, since $(x_n, x_{n+1}) \in \mathcal{P}_{N^n}(x_{n-1}, x_n)$ for all $n \in \mathbb{N}$, there exists $w_n \in \mathcal{H}(x_n)$ such that $x_{n+1} = \int_0^1 w_n(t,s)\, ds$ and

$$\|w_{n-1}(t,s) - w_n(t,s)\| \le \phi(t,s) p_{N^n}(x_{n-1}, x_n) \quad \text{a.e. } s \in [0,1], \forall t \in [0,1].$$

So,

$$\|w_n(t,s)\| \le \|w_0(t,s)\| + \phi(t,s) \sum_{n=1}^{\infty} p_{N^n}(x_{n-1}, x_n) \quad \text{a.e. } s \in [0,1], \forall t \in [0,1].$$

It follows from the Lebesgue dominated convergence theorem that there exists $w : [0,1] \times [0,1] \to E$ such that $w(t,\cdot) \in L^1([0,1],E)$ and

$$\|w(t,\cdot) - w_n(t,\cdot)\|_1 \to 0 \quad \forall t \in [0,1].$$

Also,

$$x(t) = \lim_{n\to\infty} x_n(t) = \lim_{n\to\infty} \int_0^1 w_n(t,s)\,ds = \int_0^1 w(t,s)\,ds.$$

So, $w \in \mathcal{E}$. Assumption (iii) implies that $w \in \mathcal{H}(x)$, and hence $x \in F(x)$.

Finally, Theorem 2.2.4 gives the conclusion. □

Remark 2.3.1. *A multivalued map $T : E_j \to E_j$ is said to be*

- *nondecreasing upward (resp. downward) if for every $x_j \in E_j$, $v \in T(x_j)$ and every $\hat{x}_j \in E_j$ such that $x_j \preceq \hat{x}_j$ (resp. $\hat{x}_j \preceq x_j$), there exists $\hat{v} \in T(\hat{x}_j)$ such that $v \preceq \hat{v}$ (resp. $\hat{v} \preceq v$);*
- *nonincreasing upward (resp. downward) if for every $x_j \in E_j$, $v \in T(x_j)$ and every $\hat{x}_j \in E_j$ such that $x_j \preceq \hat{x}_j$ (resp. $\hat{x}_j \preceq x_j$), there exists $\hat{v} \in T(\hat{x}_j)$ such that $\hat{v} \preceq v$ (resp. $v \preceq \hat{v}$);*
- *nondecreasing (resp. nonincreasing), if it is nondecreasing (resp. nonincreasing) upward and downward.*

Condition (ii) of the previous theorem implies that H_i is nondecreasing (resp. nonincreasing) with respect to x_j if $\sigma_j(i) = 1$ (resp. $\sigma_j(i) = -1$).

Remark 2.3.2. *Observe that Condition (iii) in the previous theorem is satisfied if $x \mapsto H(t,s,x)$ is upper semi-continuous (i.e. $\{x \in E : H(t,s,x) \cap B \neq \emptyset\}$ is closed for every closed $B \subset E$) a.e. $s \in [0,1]$ and all $t \in [0,1]$.*

In the particular case where H is single-valued, we obtain the following result.

Corollary 2.3.1. *Let $h : [0,1] \times [0,1] \times E \to E$ be a single-valued map. Assume the following conditions hold:*

(i) For every $x \in C([0,1],E)$, $h(t,\cdot,x(\cdot)) \in L^1([0,1],E)$ for all $t \in [0,1]$, and $t \mapsto \int_0^1 h(t,s,x(s))\,ds$ is continuous.

(ii) There exist $\psi \in C([0,1],E)$ such that for $i = 1,\dots,N$,

$$\psi_i(t) \preceq \int_0^1 h_i(t,s,\psi(s))\,ds \quad \forall t \in [0,1],$$

$$\text{or} \qquad \int_0^1 h_i(t,s,\psi(s))\,ds \preceq \psi_i(t) \quad \forall t \in [0,1].$$

(iii) For, $i,j = 1,\dots,N$, the map $x_j \mapsto h_i(t,s,x_1,\dots,x_{j-1},x_j,x_{j+1},\dots,x_N)$ is nondecreasing a.e. $s \in [0,1]$ and all $t \in [0,1]$, or it is nonincreasing a.e. $s \in [0,1]$ and all $t \in [0,1]$.

(iv) *There exists* $\phi : [0,1] \times [0,1] \to [0,\infty[$ *such that*

$$\phi(t,\cdot) \in L^1([0,1]) \ \forall t \in [0,1] \quad \text{and} \quad \sup_{t\in[0,1]} \|\phi(t,\cdot)\|_1 < 1,$$

and, for every $x, \hat{x} \in C([0,1],E)$ *such that* $x_i = \hat{x}_i$ *for* $i \neq j$ *and* $x_j(s) \preceq \hat{x}_j(s)$ *for all* $s \in [0,1]$, *one has*

$$\|h(t,s,x(s)) - h(t,s,\hat{x}(s))\| \leq \phi(t,s)\|x-\hat{x}\|_0.$$

(v) *For every* $x, x_n \in C([0,1],E)$, *and* $\hat{h} : [0,1]\times[0,1] \to E$, *one has* $\hat{h}(t,s) = h(t,s,x(s))$ *a.e.* $s \in [0,1]$ *and all* $t \in [0,1]$, *if*

(a) $x_n \to x$ *and* $x_{n+1}(t) = \int_0^1 h(t,s,x_n(s))\,ds$ *for all* $t \in [0,1]$ *and all* $n \in \mathbb{N}$;

(b) $h(t,s,x_n(s)) \to \hat{h}(t,s)$ *and* $\|h(t,s,x_n(s))\| \leq M\phi(t,s) + \|h(t,s,\psi(s))\|$ *a.e.* $s \in [0,1]$, *all* $t \in [0,1]$, *all* $n \in \mathbb{N}$, *and for some* $M \geq 0$.

Then, the following system of Hammerstein integral equations has a solution:

$$x(t) = \int_0^1 h(t,s,x(s))\,ds \quad \forall t \in [0,1]. \tag{2.3.4}$$

We present some corollaries of Theorem 2.3.1.

Corollary 2.3.2. *Let* $K : [0,1] \times E \to E$ *be a multivalued map with nonempty values and* $g : [0,1]\times[0,1] \to [0,\infty[^N$ *a single valued map. Let* $p \in [1,\infty]$ *and* q *its conjugate. Assume the following conditions hold:*

(i) *For every* $t \in [0,1]$, $g(t,\cdot) \in L^q([0,1],\mathbb{R}^N)$ *and* $t \mapsto g(t,\cdot)$ *is continuous.*

(ii) *There exist* $\psi \in C([0,1],E)$ *and* $\mu \in L^p([0,1],E)$ *such that* $\mu(s) \in K(s,\psi(s))$ *a.e.* $s \in [0,1]$, *and for every* $i = 1,\dots,N$

$$\psi_i(t) \preceq \int_0^1 g_i(t,s)\mu_i(s)\,ds \quad \forall t \in [0,1],$$

$$\text{or} \quad \int_0^1 g_i(t,s)\mu_i(s)\,ds \preceq \psi_i(t) \quad \forall t \in [0,1].$$

(iii) *There exists* $l \in L^p([0,1],[0,\infty[^N)$ *such that*

$$\alpha = \sup_{t\in[0,1]} \max\{\|l_1(\cdot)g_1(t,\cdot)\|_1, \dots, \|l_N(\cdot)g_N(t,\cdot)\|_1\} < 1,$$

and for $j = 1,\dots,N$, *there exists* $\sigma_j : \{1,\dots,N\} \to \{1,-1\}$ *such that, for every* $x \in C([0,1],E)$ *and every* $k \in L^p([0,1],E)$ *such that* $k(s) \in K(s,x(s))$ *a.e.* $s \in [0,1]$, *one has, for every* $\hat{x} \in C([0,1],E)$ *such that* $x_i = \hat{x}_i$ *for* $i \neq j$ *and* $x_j(s) \preceq \hat{x}_j(s)$ *for all* $s \in [0,1]$ *(resp.* $\hat{x}_j(s) \preceq x_j(s)$ *for all* $s \in [0,1]$*), there exists* $\hat{k} \in L^p([0,1],E)$ *such that, a.e.* $s \in [0,1]$, $\hat{k}(s) \in K(s,\hat{x}(s))$,

$$\|\hat{k}_i(s) - k_i(s)\| \leq l_i(s)\|\hat{x}_j - x_j\|_0,$$

and

$$0 \preceq \sigma_j(i)\Big(\hat{k}_i(s) - k_i(s)\Big) \quad \forall i = 1, \ldots, N;$$
$$\textit{(resp.} \quad 0 \preceq \sigma_j(i)\Big(k_i(s) - \hat{k}_i(s)\Big) \quad \forall i = 1, \ldots, N).$$

(iv) For every $x, x_n \in C([0,1], E)$ and for every $k, k_n \in L^p([0,1], E)$ such that $k_n(s) \in K(s, x_n(s))$ a.e. $s \in [0,1]$, one has $k(s) \in K(s, x(s))$ a.e. $s \in [0,1]$ if

(a) $x_n \to x$ and $x_{n,i}(t) = \int_0^1 g_i(t,s) k_{n-1,i}(s)\, ds$ for all $t \in [0,1]$, $n \in \mathbb{N}$, and $i = 1, \ldots, N$;

(b) $k_n(s) \to k(s)$ and $|k_{n,i}(s)| \leq M l_i(s) + |k_{0,i}(s)|$ a.e. $s \in [0,1]$, all $i = 1, \ldots, N$, all $n \in \mathbb{N}$ and for some $M \geq 0$.

Then, the following system has a solution:

$$x_i(t) \in \int_0^1 g_i(t,s) K_i(s, x(s))\, ds \quad \forall t \in [0,1],\ i = 1, \ldots, N. \tag{2.3.5}$$

Now, we consider the initial value problem for a system of differential inclusions:

$$\begin{aligned} &x'(t) \in K(t, x(t)) \quad \text{a.e. } t \in [0,1], \\ &x(0) = r. \end{aligned} \tag{2.3.6}$$

Corollary 2.3.3. *Let $r \in E$ and $K : [0,1] \times E \to E$ a multivalued map with nonempty values. Assume the following conditions hold:*

(i) There exist $\psi \in C([0,1], E)$, $\nu \in L^1([0,1], E)$ such that $\nu(s) \in K(s, \psi(s))$ a.e. $s \in [0,1]$, and for all $i = 1, \ldots, N$,

$$\psi_i(t) \preceq r_i + \int_0^t \nu_i(s)\, ds \quad \forall t \in [0,1],$$
$$\textit{or} \quad r_i + \int_0^t \nu_i(s)\, ds \preceq \psi_i(t) \quad \forall t \in [0,1].$$

(ii) There exists $l \in L^1([0,1])$ such that $\|l\|_1 < 1$ and for $j = 1, \ldots, N$, there exists $\sigma_j : \{1, \ldots, N\} \to \{1, -1\}$ such that for every $x \in C([0,1], E)$ and every $k \in L^1([0,1], E)$ such that $k(s) \in K(s, x(s))$ a.e. $s \in [0,1]$, one has, for every $\hat{x} \in C([0,1], E)$ such that $x_i = \hat{x}_i$ for $i \neq j$, $x_j(s) \preceq \hat{x}_j(s)$ for all $s \in [0,1]$, (resp. $\hat{x}_j(s) \preceq x_j(s)$ for all $s \in [0,1]$), there exists $\hat{k} \in L^1([0,1], E)$ such that a.e. $s \in [0,1]$, $\hat{k}(s) \in K(s, \hat{x}(s))$,

$$\|\hat{k}_i(s) - k_i(s)\| \leq l(s) \|\hat{x} - x\|_0,$$

and

$$0 \preceq \sigma_j(i)\Big(\hat{k}_i(s) - k_i(s)\Big), \quad \forall i = 1, \ldots, N,$$
$$\textit{(resp.} \quad 0 \preceq \sigma_j(i)\Big(k_i(s) - \hat{k}_i(s)\Big), \quad \forall i = 1, \ldots, N).$$

(iii) *For every* $x \in C([0,1],E)$*,* $x_n \in W^{1,1}([0,1],E)$ *and every* $k \in L^1([0,1],E)$ *such that* $x_n(0) = r$*,* $x'_{n+1}(s) \in K(s, x_n(s))$ *a.e.* $s \in [0,1]$*, one has* $k(s) \in K(s,x(s))$ *a.e.* $s \in [0,1]$ *if* $\|x_n - x\|_0 \to 0$*,* $x'_n(s) \to k(s)$ *a.e.* $s \in [0,1]$ *and*

$$\|x'_n(s)\| \leq Ml(s) + \|\nu(s)\| \quad \text{a.e. } s \in [0,1], \ \forall n \in \mathbb{N} \quad \text{for some } M \geq 0.$$

Then, (2.3.6) has a solution.

Proof. Let $H : [0,1] \times [0,1] \times E \to E$ be given by

$$H(t,s,x) = r + \chi_{[0,t]}(s)K(s,x).$$

The conclusion follows from Theorem 2.3.1. □

Remark 2.3.3. *Observe that (i) of the previous corollary is satisfied if there exist* $\sigma : \{1,\dots,N\} \to \{1,-1\}$*,* $\psi \in W^{1,1}([0,1],E)$ *and* $\nu \in L^1([0,1],E)$ *such that* $\nu(s) \in K(s,\psi(s))$ *a.e.* $s \in [0,1]$*, and, for all* $i = 1,\dots,N$*,*

$$\begin{aligned} &\text{if } \sigma(i) = 1, && \psi_i(0) \preceq r_i, \quad \psi'_i(s) \preceq \nu_i(s) \quad \text{a.e. } s \in [0,1],\\ &\text{if } \sigma(i) = -1, && r_i \preceq \psi_i(0), \quad \nu_i(s) \preceq \psi'_i(s) \quad \text{a.e. } s \in [0,1]. \end{aligned}$$

In the particular case where $N = 1$ *and* $E_1 = \mathbb{R}$*, such a function* ψ *is called a lower solution (resp. upper solution) of (2.3.6) if* $\sigma(1) = 1$ *(resp.* $\sigma(1) = -1$*).*

We consider the periodic boundary value problem for a system of differential inclusions:

$$\begin{aligned} &x'(t) \in K(t,x(t)) \quad \text{a.e. } t \in [0,1],\\ &x(0) = x(1). \end{aligned} \tag{2.3.7}$$

Corollary 2.3.4. *Let* $K : [0,1] \times E \to E$ *be a multivalued map with nonempty values. Assume the following conditions hold:*

(i) *There exist* $\psi \in W^{1,1}([0,1],E)$ *and* $\nu \in L^1([0,1],E)$ *such that* $\nu(s) \in K(s,\psi(s))$ *a.e.* $s \in [0,1]$*, and, for all* $i = 1,\dots,N$*,*

$$\begin{aligned} &\psi_i(0) \preceq \psi(1) \quad \text{and} \quad \psi'_i(s) \preceq \nu_i(s) \quad \text{a.e. } s \in [0,1],\\ \text{or} \quad &\psi_i(1) \preceq \psi(0) \quad \text{and} \quad \nu_i(s) \preceq \psi'_i(s) \quad \text{a.e. } s \in [0,1]. \end{aligned}$$

(ii) *There exist* $0 \leq l < a$ *and for* $j = 1,\dots,N$*, there exists* $\sigma_j : \{1,\dots,N\} \to \{1,-1\}$ *such that for every* $x \in C([0,1],E)$ *and every* $k \in L^1([0,1],E)$ *with* $k(s) \in K(s,x(s))$ *a.e.* $s \in [0,1]$*, one has, for every* $\hat{x} \in C([0,1],E)$ *such that* $x_i = \hat{x}_i$ *for all* $i \neq j$ *and* $x_j(s) \preceq \hat{x}_j(s)$ *for all* $s \in [0,1]$*, (resp.* $\hat{x}_j(s) \preceq x_j(s)$ *for all* $s \in [0,1]$*), there exists* $\hat{k} \in L^1([0,1],E)$ *such that*

a.e. $s \in [0,1]$, $\hat{k}(s) \in K(s, \hat{x}(s))$,

$$\|\hat{k}_i(s) + a\hat{x}_i(s) - k_i(s) - ax_i(s)\| \le l\|\hat{x} - x\|_0,$$

and

$$0 \preceq \sigma_j(i)\Big(\hat{k}_i(s) + a\hat{x}_i(s) - k_i(s) - ax_i(s)\Big) \quad \forall i = 1, \dots, N,$$

$$\text{(resp.}\quad 0 \preceq \sigma_j(i)\Big(k_i(s) + ax_i(s) - \hat{k}_i(s) - a\hat{x}_i(s)\Big) \quad \forall i = 1, \dots, N).$$

(iii) For every $x \in C([0,1], E)$, $x_n \in W^{1,1}([0,1], E)$ and every $k \in L^1([0,1], E)$ such that $x_n(0) = x_n(1)$, $x'_{n+1}(s) \in K(s, x_n(s)) + a(x_n(s) - x_{n+1}(s))$ a.e. $s \in [0,1]$, one has $k(s) \in K(s, x(s))$ a.e. $s \in [0,1]$ if $\|x_n - x\|_0 \to 0$, $x'_n(s) \to k(s)$ a.e. $s \in [0,1]$ and

$$\|x'_n(s)\| \le Ml + \|\nu(s)\| \quad \text{a.e. } s \in [0,1],\ \forall n \in \mathbb{N} \quad \text{for some } M \ge 0.$$

Then, (2.3.7) has a solution.

Proof. Let $g : [0,1] \times [0,1] \to [0, \infty[$ be defined by

$$g(t,s) = \begin{cases} \frac{e^{a(s-t+1)}}{e^a - 1}, & \text{if } 0 \le s \le t \le 1, \\ \frac{e^{a(s-t)}}{e^a - 1}, & \text{if } 0 \le t < s \le 1. \end{cases}$$

A solution of (2.3.7) is a solution of

$$x_i(t) \in \int_0^1 g(t,s)\Big(K_i(s, x(s)) + ax_i(s)\Big)\, ds \quad \forall t \in [0,1],\ i = 1, \dots, N.$$

The conclusion follows from Corollary 2.3.2. □

Remark 2.3.4. *In the particular case where $N = 1$ and $E_1 = \mathbb{R}$, the function ψ satisfying Condition (i) of Corollary 2.3.4 is called a lower solution (resp. upper solution) of (2.3.7) if $\sigma(1) = 1$ (resp. $\sigma(1) = -1$).*

Remark 2.3.5. *In the previous corollary, we can replace $l < a$ by $l \in L^1([0,1])$ such that*

$$\sup_{t \in [0,1]} \int_0^1 g(t,s) l(s)\, ds < 1.$$

2.4. Hammerstein integral inclusions with nonincreasing type conditions

In the previous section, we established the existence of a solution to the system (2.1.1), where all H_i satisfy monotonicity type conditions with respect to each variable x_j. In this section, we consider a particular case where H satisfies a nonincreasing type condition. In this particular case, the continuity condition (see Condition (iii) of Theorem 2.3.1) can be weaken. Also, the use of weak G-contraction will not be necessary in the proof since the associated operator will be a G-contraction.

Theorem 2.4.1. *Let $H : [0,1] \times [0,1] \times E \to E$ be a multivalued map with nonempty values. Assume the following conditions hold:*

(i) There exist $x_0 \in C([0,1],E)$, $w_0 \in \mathcal{H}(x_0)$ and $\sigma : \{1,\dots,N\} \to \{1,-1\}$ such that

$$0 \preceq \sigma(i)\left(\int_0^1 w_{0,i}(t,s)\,ds - x_{0,i}(t)\right) \quad \forall t \in [0,1],\ \forall i = 1,\dots,N.$$

(ii) There exists $\phi : [0,1] \times [0,1] \to [0,\infty[$ such that

$$\phi(t,\cdot) \in L^1([0,1])\ \forall t \in [0,1] \quad \textit{and} \quad \sup_{t\in[0,1]} \|\phi(t,\cdot)\|_1 < 1,$$

and, for every $x \in C([0,1],E)$, $w \in \mathcal{H}(x)$ and every $\hat{x} \in C([0,1],E)$ such that

$$0 \preceq \sigma(i)\Big(\hat{x}_i(s) - x_i(s)\Big) \quad \forall s \in [0,1],\ \forall i = 1,\dots,N,$$

$$\textit{(resp.}\quad 0 \preceq \sigma(i)\Big(x_i(s) - \hat{x}_i(s)\Big) \quad \forall s \in [0,1],\ \forall i = 1,\dots,N),$$

there exists $\hat{w} \in \mathcal{H}(\hat{x})$ such that, a.e. $s \in [0,1]$ and all $t \in [0,1]$,

$$\|w(t,s) - \hat{w}(t,s)\| \le \phi(t,s)\|x - \hat{x}\|_0,$$

and

$$0 \preceq \sigma(i)(w_i(t,s) - \hat{w}_i(t,s)) \quad \forall i = 1,\dots,N,$$

$$\textit{(resp.}\quad 0 \preceq \sigma(i)(\hat{w}_i(t,s) - \hat{w}_i(t,s)) \quad \forall i = 1,\dots,N).$$

(iii) For every $x, x_n \in C([0,1],E)$, $w \in \mathcal{E}$ and $w_n \in \mathcal{H}(x_n)$ such that

$$\sum_{n=1}^{\infty} \|x_n - x_{n+1}\|_0 < \infty, \quad x_{n+1}(t) = \int_0^1 w_n(t,s)\,ds,$$

and

$$\|w_n(t,s) - w_{n+1}(t,s)\| \le \phi(t,s)\|x_n - x_{n+1}\|_0 \qquad \textit{a.e. } s \in [0,1], \forall t \in [0,1], \forall n \in \mathbb{N},$$

one has $w \in \mathcal{H}(x)$, where $\|x_n - x\|_0 \to 0$ and $\|w_n(t,\cdot) - w(t,\cdot)\|_1 \to 0$.

Then, (2.1.1) has a solution.

Proof. We consider on $C([0,1],E)$ the graph G^r with $V(G^r) = C([0,1],E)$ and (E(G^r)) one has $(x,y) \in E(G^r)$ if and only if one of the following conditions hold:

(a) $0 \preceq \sigma(i)\Big(\hat{x}_i(s) - x_i(s)\Big)$ for all $s \in [0,1]$ and all $i = 1, \ldots, N$;

(b) $0 \preceq \sigma(i)\Big(x_i(s) - \hat{x}_i(s)\Big)$ for all $s \in [0,1]$ and all $i = 1, \ldots, N$.

Let
$$x_1(t) = \int_0^1 w_0(t,s)\,ds \quad \forall t \in [0,1].$$
It follows from (i) that $(x_0, x_1) \in E(G^r)$ and $x_1 \in F(x_0)$.

We consider the following property:

(P^r) For $(x, \hat{x}) \in E(G^r)$, we say that $(u, \hat{u}) \in \mathcal{P}^r(x, \hat{x})$ if for all $w \in \mathcal{H}(x)$ such that $u(t) = \int_0^1 w(t,s)\,ds$, there exists $\hat{w} \in \mathcal{H}(\hat{x})$ such that

(a) $\hat{u}(t) = \int_0^1 \hat{w}(t,s)\,ds$;

(b) $\|w(t,s) - \hat{w}(t,s)\| \leq \phi(t,s)\|x - \hat{x}\|_0$ a.e. $s \in [0,1]$, and all $t \in [0,1]$.

Let F be the associated multivalued map defined in (2.3.3). It follows from (ii) that F satisfies Condition (ii) of Theorem 2.2.2 with $m = 1$.

Finally, Condition (iii) of Theorem 2.2.2 follows from Assumption (iii). $\square$

Remark 2.4.1. *If H satisfies (i) and (ii) of Theorem 2.3.1 with $\sigma_0, \ldots, \sigma_N$ such that*
$$\sigma_0(j)\sigma_j(i) = -\sigma_0(i) \quad \forall i, j = 1, \ldots, N, \tag{2.4.1}$$
then $\sigma = \sigma_0$ satisfies (i) and (ii) of the previous theorem. Indeed, without loss of generality, let $x, \hat{x} \in C([0,1], E)$ be such that
$$0 \preceq \sigma(i)(\hat{x}_i(s) - x_i(s)) \quad \forall s \in [0,1],\ \forall i = 1, \ldots, N.$$
Let $w \in \mathcal{H}(x)$. By (ii) of Theorem 2.3.1, there exists $\hat{w}^1 \in \mathcal{H}(\hat{x}_1, x_2, \ldots, x_N)$ such that
$$\|w(t,s) - \hat{w}^1(t,s)\| \leq \phi(t,s)\|x_1 - \hat{x}_1\|_0,$$
and
$$\begin{aligned} &\text{if } \sigma(1) = 1, && 0 \preceq \sigma_1(i)(\hat{w}_i^1(t,s) - w_i(t,s)) \quad \forall i = 1, \ldots, N, \\ &\text{if } \sigma(1) = -1, && 0 \preceq \sigma_1(i)(w_i(t,s) - \hat{w}_i^1(t,s)) \quad \forall i = 1, \ldots, N. \end{aligned}$$
So, by (2.4.1),
$$0 \preceq \sigma(i)(w_i(t,s) - \hat{w}_i^1(t,s)) = \sigma(1)\sigma_1(i)(\hat{w}_i^1(t,s) - w_i(t,s)) \quad \forall i = 1, \ldots, N.$$
By the same argument, for $j = 2, \ldots, N$, there exists $\hat{w}^j \in \mathcal{H}(\hat{x}_1, \ldots, \hat{x}_j, x_{j+1}, \ldots, x_N)$ such that
$$\|\hat{w}^{j-1}(t,s) - \hat{w}^j(t,s)\| \leq \phi(t,s)\|x_j - \hat{x}_j\|_0,$$
and
$$0 \preceq \sigma(i)(\hat{w}_i^{j-1}(t,s) - \hat{w}_i^j(t,s)) = \sigma(j)\sigma_j(i)(\hat{w}_i^j(t,s) - \hat{w}_i^{j-1}(t,s)) \quad \forall i = 1, \ldots, N.$$

Therefore, $\hat{w} = \hat{w}^N \in \mathcal{H}(\hat{x})$ *is such that*

$$\|w(t,s) - \hat{w}(t,s)\| \leq \sum_{j=1}^{N} \|\hat{w}^{j-1}(t,s) - \hat{w}^{j}(t,s)\| \leq \phi(t,s) \sum_{j=1}^{N} \|x_j - \hat{x}_j\|_0 = \phi(t,s)\|x - \hat{x}\|_0,$$

and

$$0 \preceq \sigma(i)(w_i(t,s) - \hat{w}_i(t,s)) = \sum_{j=1}^{N} \sigma(i)(\hat{w}_i^{j-1}(t,s) - \hat{w}_i^{j}(t,s)) \quad \forall i = 1, \ldots, N.$$

As in the previous section, we obtain as corollaries existence results for systems of differential inclusions with initial value condition or periodic boundary value condition.

Corollary 2.4.1. *Let* $r \in E$ *and* $K : [0,1] \times E \to E$ *a multivalued map with nonempty values. Assume the following conditions hold:*

(i) There exist $\sigma : \{1, \ldots, N\} \to \{1, -1\}$, $\psi \in C([0,1], E)$, $\nu \in L^1([0,1], E)$ *such that* $\nu(s) \in K(s, \psi(s))$ *a.e.* $s \in [0,1]$, *and*

$$\sigma(i) \left(\psi_i(t) - r_i - \int_0^t \nu_i(s)\, ds \right) \preceq 0 \quad \forall t \in [0,1],\ \forall i = 1, \ldots, N.$$

(ii) There exists $l \in L^1([0,1])$ *such that* $\|l\|_1 < 1$ *and for every* $x \in C([0,1], E)$ *and every* $k \in L^1([0,1], E)$ *such that* $k(s) \in K(s, x(s))$ *a.e.* $s \in [0,1]$, *one has, for every* $\hat{x} \in C([0,1], E)$ *such that*

$$0 \preceq \sigma(i)\Big(\hat{x}_i(s) - x_i(s)\Big) \quad \forall s \in [0,1],\ \forall i = 1, \ldots, N,$$
$$\textit{(resp.}\quad 0 \preceq \sigma(i)\Big(x_i(s) - \hat{x}_i(s)\Big) \quad \forall s \in [0,1],\ \forall i = 1, \ldots, N),$$

there exists $\hat{k} \in L^1([0,1], E)$ *such that a.e.* $s \in [0,1]$, $\hat{k}(s) \in K(s, \hat{x}(s))$, $\|k(s) - \hat{k}(s)\| \leq l(s)\|\hat{x} - x\|_0$, *and*

$$0 \preceq \sigma(i)\Big(k_i(s) - \hat{k}_i(s)\Big), \quad \forall i = 1, \ldots, N,$$
$$\textit{(resp.}\quad 0 \preceq \sigma(i)\Big(\hat{k}_i(s) - k_i(s)\Big), \quad \forall i = 1, \ldots, N).$$

(iii) For every $x_n \in W^{1,1}([0,1], E)$ *such that* $x_n(0) = r$, $x'_{n+1}(t) \in K(s, x_n(s))$ *a.e.* $s \in [0,1]$,

$$\sum_{n=1}^{\infty} \|x_n - x_{n+1}\|_0 < \infty,$$

and

$$\|x'_n(s) - x'_{n+1}(s)\| \leq l(s)\|x_{n-1} - x_n\|_0 \quad \textit{a.e. } s \in [0,1],\ \forall n \in \mathbb{N};$$

one has $k(s) \in K(s, x(s))$ *a.e.* $s \in [0,1]$, *where* $\|x_n - x\|_0 \to 0$ *and* $\|x'_n - k\|_1 \to 0$.

Then, (2.3.6) *has a solution.*

Corollary 2.4.2. *Let* $K : [0,1] \times E \to E$ *be a multivalued map with nonempty values. Assume the following conditions hold:*

(i) There exist $\sigma : \{1,\ldots,N\} \to \{1,-1\}$, $\psi \in W^{1,1}([0,1],E)$ *and* $\nu \in L^1([0,1],E)$ *such that* $\nu(s) \in K(s,\psi(s))$ *a.e.* $s \in [0,1]$, *and, for all* $i = 1,\ldots,N$,

$$\sigma(i)\Big(\psi_i(0) - \psi(1)\big) \preceq 0 \quad \textit{and} \quad \sigma(i)\Big(\psi_i'(s) - \nu_i(s)\Big) \preceq 0 \quad \textit{a.e. } s \in [0,1].$$

(ii) There exist $0 \leq l < a$ *such that for every* $x \in C([0,1],E)$ *and* $k \in L^1([0,1],E)$ *with* $k(s) \in K(s,x(s))$ *a.e.* $s \in [0,1]$, *one has, for every* $\hat{x} \in C([0,1],E)$ *such that*

$$\begin{aligned} 0 \preceq \sigma(i)\Big(\hat{x}_i(s) - x_i(s)\Big) \quad \forall s \in [0,1],\ \forall i = 1,\ldots,N, \\ \textit{(resp.}\quad 0 \preceq \sigma(i)\Big(x_i(s) - \hat{x}_i(s)\Big) \quad \forall s \in [0,1],\ \forall i = 1,\ldots,N), \end{aligned}$$

there exists $\hat{k} \in L^1([0,1],E)$ *such that a.e.* $s \in [0,1]$, $\hat{k}(s) \in K(s,\hat{x}(s))$,

$$\|k(s) + ax(s) - \hat{k}(s) - a\hat{x}(s)\| \leq l\|\hat{x} - x\|_0,$$

and

$$\begin{aligned} 0 \preceq \sigma(i)\Big(k_i(s) + ax_i(s) - \hat{k}_i(s) - a\hat{x}_i(s)\Big) \quad \forall i = 1,\ldots,N, \\ \textit{(resp.}\quad 0 \preceq \sigma(i)\Big(\hat{k}_i(s) + a\hat{x}_i(s) - k_i(s) - ax_i(s)\Big) \quad \forall i = 1,\ldots,N). \end{aligned}$$

(iii) For every $x_n \in W^{1,1}([0,1],E)$ *such that* $x_n(0) = x_n(1)$,

$$\sum_{n=1}^{\infty} \|x_n - x_{n+1}\|_0 < \infty, \quad x_{n+1}'(s) \in K(s,x_n(s)) + a(x_n(s) - x_{n+1}(s)),$$

and

$$\|x_n'(s) - x_{n+1}'(s) + ax_{n-1}(s) - ax_n(s)\| \leq l\|x_{n-1} - x_n\|_0 \qquad \textit{a.e. } s \in [0,1],\ \forall n \in \mathbb{N};$$

one has $k(s) \in K(s,x(s))$ *a.e.* $s \in [0,1]$, *where* $\|x_n - x\|_0 \to 0$ *and* $\|x_n' - k\|_1 \to 0$.

Then, (2.3.7) *has a solution.*

Remark 2.4.2. *In the particular case where* $N = 1$, $E_1 = \mathbb{R}$ *and* K *is a continuous single-valued map, the previous corollary is due to Nieto and Rodríguez-López [37].*

2.5. Hammerstein integral inclusions with nondecreasing type conditions

In the previous section, we considered the particular case where H satisfies a nonincreasing type condition. In this section, we establish the existence of a solution to the system (2.1.1) in the particular case where H satisfies a nondecreasing (upward or downward) type condition.

Theorem 2.5.1. *Let $H : [0,1] \times [0,1] \times E \to E$ be a multivalued map with nonempty values. Assume the following conditions hold:*

(i) There exist $x_0 \in C([0,1], E)$, $w_0 \in \mathcal{H}(x_0)$ and $\sigma : \{1, \ldots, N\} \to \{1, -1\}$ such that

$$0 \preceq \sigma(i) \left(\int_0^1 w_{0,i}(t,s)\, ds - x_{0,i}(t) \right) \quad \forall t \in [0,1],\ \forall i = 1, \ldots, N.$$

(ii) There exists $\phi : [0,1] \times [0,1] \to [0, \infty[$ such that

$$\phi(t, \cdot) \in L^1([0,1])\ \forall t \in [0,1] \quad \textit{and} \quad \sup_{t \in [0,1]} \|\phi(t, \cdot)\|_1 < 1,$$

and, for every $x \in C([0,1], E)$, $w \in \mathcal{H}(x)$ and every $\hat{x} \in C([0,1], E)$ such that

$$0 \preceq \sigma(i) \Big(\hat{x}_i(s) - x_i(s) \Big) \quad \forall s \in [0,1],\ \forall i = 1, \ldots, N,$$

there exists $\hat{w} \in \mathcal{H}(\hat{x})$ such that, a.e. $s \in [0,1]$ and all $t \in [0,1]$,

$$\|w(t,s) - \hat{w}(t,s)\| \leq \phi(t,s) \|x - \hat{x}\|_0,$$

and

$$0 \preceq \sigma(i) (\hat{w}_i(t,s) - w_i(t,s)) \quad \forall i = 1, \ldots, N.$$

(iii) For every $x, x_n \in C([0,1], E)$, $w \in \mathcal{E}$ and $w_n \in \mathcal{H}(x_n)$ such that

$$\sum_{n=1}^{\infty} \|x_n - x_{n+1}\|_0 < \infty, \quad x_{n+1}(t) = \int_0^1 w_n(t,s)\, ds$$

and

$$\|w_n(t,s) - w_{n+1}(t,s)\| \leq \phi(t,s) \|x_n - x_{n+1}\|_0 \qquad \textit{a.e. } s \in [0,1], \forall t \in [0,1], \forall n \in \mathbb{N},$$

one has $w \in \mathcal{H}(x)$, where $\|x_n - x\|_0 \to 0$ and $\|w_n(t, \cdot) - w(t, \cdot)\|_1 \to 0$.

Then, (2.1.1) has a solution.

Proof. We consider on $C([0,1], E)$ the graph G^d with $V(G^d) = C([0,1], E)$ and

(E(G^d)) one has $(x,y)\in E(G^d)$ if and only if $0\preceq \sigma(i)\Big(\hat{x}_i(s)-x_i(s)\Big)$ for all $s\in[0,1]$ and all $i=1,\dots,N$.

We consider the following property:

(P^d) For $(x,\hat{x})\in E(G^d)$, we say that $(u,\hat{u})\in\mathcal{P}^d(x,\hat{x})$ if for all $w\in\mathcal{H}(x)$ such that $u(t)=\int_0^1 w(t,s)\,ds$, there exists $\hat{w}\in\mathcal{H}(\hat{x})$ such that

(a) $\hat{u}(t)=\int_0^1 \hat{w}(t,s)\,ds$;

(b) $\|w(t,s)-\hat{w}(t,s)\|\le\phi(t,s)\|x-\hat{x}\|_0$ a.e. $s\in[0,1]$, and all $t\in[0,1]$;

(c) $0\preceq\sigma(i)\Big(\hat{w}_i(t,s)-w_i(t,s)\Big)$ for a.e. $s\in[0,1]$, all $t\in[0,1]$ and all $i=1,\dots,N$.

It can be shown that the operator F defined in (2.3.3) satisfies the assumptions of Theorem 2.2.2. □

In the next result, we show that the continuity type condition (iii) of the previous result can be removed if one assumes that H has closed values and if the order on the Banach spaces E_i satisfies an additional property.

Theorem 2.5.2. *Let $H:[0,1]\times[0,1]\times E\to E$ be a multivalued map with nonempty closed values. Assume that Assumptions (i) and (ii) of Theorem 2.5.1 are satisfied. In addition, assume that the following conditions hold:*

(iii)' For every $i=1,\dots,N$, and every sequence $\{a_n\}$ in E_i such that $a_n\to a$ and $0\preceq\sigma(i)(a_{n+1}-a_n)$ for all $n\in\mathbb{N}$, one has $0\preceq\sigma(i)(a-a_n)$ for all $n\in\mathbb{N}$.

Then, (2.1.1) has a solution.

Proof. Let G^d and $\mathcal{P}^d(x,\hat{x})$ be the graph and the property introduced in the proof of the previous theorem. As in its proof, it follows from (i) and (ii) that Assumptions (i) and (ii) of Theorem 2.2.2 are satisfied.

We claim that F satisfies (iii) of Theorem 2.2.2 with $m=1$. Let $\{x_n\}$ in $T_1(F,G^d,x_0)$ be such that $(x_n,x_{n+1})\in\mathcal{P}^d(x_{n-1},x_n)$ for all $n\in\mathbb{N}$ and

$$\sum_{n=1}^{\infty}\|x_{n-1}-x_n\|_0<\infty.$$

This a Cauchy sequence converging to some $x\in C([0,1],E)$. Since $(x_n,x_{n+1})\in\mathcal{P}^d(x_{n-1},x_n)$ for all $n\in\mathbb{N}$, there exists $w_n\in\mathcal{H}(x_n)$ such that $x_{n+1}=\int_0^1 w_n(t,s)\,ds$,

$$\|w_{n-1}(t,s)-w_n(t,s)\|\le\phi(t,s)\|x_{n-1}-x_n\|_0,$$

and

$$0\preceq\sigma(i)\Big(w_{n,i}(t,s)-w_{n-1,i}(t,s)\Big)$$

$$\text{a.e. } s \in [0,1], \forall t \in [0,1], \forall i = 1, \ldots, N, \forall n \in \mathbb{N}.$$

Also,

$$\|w_n(t,s)\| \leq \|w_0(t,s)\| + \phi(t,s) \sum_{n=1}^{\infty} \|x_{n-1} - x_n\|_0 \quad \text{a.e. } s \in [0,1], \forall t \in [0,1].$$

It follows from the Lebesgue dominated convergence theorem that there exists $w : [0,1] \times [0,1] \to E$ such that $w(t,\cdot) \in L^1([0,1],E)$ and

$$\|w(t,\cdot) - w_n(t,\cdot)\|_1 \to 0 \quad \forall t \in [0,1].$$

Moreover,

$$x(t) = \lim_{n\to\infty} x_n(t) = \lim_{n\to\infty} \int_0^1 w_n(t,s)\, ds = \int_0^1 w(t,s)\, ds \quad \forall t \in [0,1].$$

So, $w \in \mathcal{E}$. Also, by (iii)', $0 \preceq \sigma(i)\Big(x_i(t) - x_{n,i}(t)\Big)$ for all $t \in [0,1]$ and all $i = 1, \ldots, N$. It follows from (ii), that for every $n \in \mathbb{N}$, there exists $\hat{w}_n \in \mathcal{H}(x)$ such that

$$\|w_n(t,s) - \hat{w}_n(t,s)\| \leq \phi(t,s)\|x_n - x\|_0,$$

and

$$0 \preceq \sigma(i)(\hat{w}_n(t,s) - w_n(t,s)) \quad \text{a.e. } s \in [0,1],\ \forall t \in [0,1],\ \forall i = 1, \ldots, N.$$

Therefore,

$$\hat{w}_n(t,s) \to w(t,s) \quad \text{a.e. } s \in [0,1],\ \forall t \in [0,1].$$

The fact that H has closed values implies that $w \in \mathcal{H}(x)$, and hence $x \in F(x)$.

Finally, Theorem 2.2.2 gives the conclusion. □

Remark 2.5.1. *Assumption (ii) of Theorems 2.5.1 and 2.5.2 can be weaken by restricting the condition to $x \in C([0,1],E)$ such that $0 \preceq \sigma(i)(x_i(s) - x_{0,i}(s))$ for all $s \in [0,1]$ and $i = 1, \ldots, N$. So, Theorem 2.5.1 and 2.5.2 hold with (ii) replaced by*

(ii)' There exists $\phi : [0,1] \times [0,1] \to [0,\infty[$ such that

$$\phi(t,\cdot) \in L^1([0,1])\ \forall t \in [0,1] \quad \textit{and} \quad \sup_{t\in[0,1]} \|\phi(t,\cdot)\|_1 < 1,$$

and, for every $x \in C([0,1],E)$, $w \in \mathcal{H}(x)$ and every $\hat{x} \in C([0,1],E)$ such that

$$0 \preceq \sigma(i)\Big(x_i(s) - x_{0,i}(s)\Big) \quad \textit{and}$$
$$0 \preceq \sigma(i)\Big(\hat{x}_i(s) - x_i(s)\Big) \quad \forall s \in [0,1],\ \forall i = 1, \ldots, N,$$

there exists $\hat{w} \in \mathcal{H}(\hat{x})$ *such that, a.e.* $s \in [0,1]$ *and all* $t \in [0,1]$,

$$\|w(t,s) - \hat{w}(t,s)\| \leq \phi(t,s)\|x - \hat{x}\|_0,$$

and

$$0 \preceq \sigma(i)(\hat{w}_i(t,s) - w_i(t,s)) \quad \forall i = 1, \ldots, N.$$

We state existence results for systems of differential inclusions with initial value condition or periodic boundary value condition which follow directly from the previous theorem.

Corollary 2.5.1. *Let* $r \in E$ *and* $K : [0,1] \times E \to E$ *a multivalued map with nonempty closed values. Assume the following conditions hold:*

(i) There exist $\sigma : \{1, \ldots, N\} \to \{1, -1\}$, $\psi \in C([0,1], E)$, $\nu \in L^1([0,1], E)$ *such that* $\nu(s) \in K(s, \psi(s))$ *a.e.* $s \in [0,1]$, *and*

$$\sigma(i)\left(\psi_i(t) - r_i - \int_0^t \nu_i(s)\, ds\right) \preceq 0 \quad \forall t \in [0,1],\ \forall i = 1, \ldots, N.$$

(ii) There exists $l \in L^1([0,1])$ *such that* $\|l\|_1 < 1$ *and for every* $x \in C([0,1], E)$ *and every* $k \in L^1([0,1], E)$ *such that* $k(s) \in K(s, x(s))$ *a.e.* $s \in [0,1]$, *one has, for every* $\hat{x} \in C([0,1], E)$ *such that*

$$0 \preceq \sigma(i)\Big(\hat{x}_i(s) - x_i(s)\Big) \quad \forall s \in [0,1],\ \forall i = 1, \ldots, N.$$

there exists $\hat{k} \in L^1([0,1], E)$ *such that a.e.* $s \in [0,1]$, $\hat{k}(s) \in K(s, \hat{x}(s))$,

$$\|k(s) - \hat{k}(s)\| \leq l(s)\|\hat{x} - x\|_0,$$

and

$$0 \preceq \sigma(i)\Big(\hat{k}_i(s) - k_i(s)\Big) \quad \forall i = 1, \ldots, N.$$

(iii) For every $i = 1, \ldots, N$, *and every sequence* $\{a_n\}$ *in* E_i *such that* $a_n \to a$ *and* $0 \preceq \sigma(i)(a_{n+1} - a_n)$ *for all* $n \in \mathbb{N}$, *one has* $0 \preceq \sigma(i)(a - a_n)$ *for all* $n \in \mathbb{N}$.

Then, (2.3.6) *has a solution.*

Corollary 2.5.2. *Let* $K : [0,1] \times E \to E$ *be a multivalued map with nonempty closed values. Assume the following conditions hold:*

(i) There exist $\sigma : \{1, \ldots, N\} \to \{1, -1\}$, $\psi \in W^{1,1}([0,1], E)$ *and* $\nu \in L^1([0,1], E)$ *such that* $\nu(s) \in K(s, \psi(s))$ *a.e.* $s \in [0,1]$, *and, for all* $i = 1, \ldots, N$,

$$\sigma(i)\Big(\psi_i(0) - \psi(1)\Big) \preceq 0 \quad \text{and} \quad \sigma(i)\Big(\psi_i'(s) - \nu_i(s)\Big) \preceq 0 \quad \text{a.e. } s \in [0,1].$$

(ii) *There exist* $0 \leq l < a$ *such that for every* $k \in L^1([0,1], E)$ *with* $k(s) \in K(s, x(s))$ *a.e.* $s \in [0,1]$*, one has, for every* $\hat{x} \in C([0,1], E)$ *such that*

$$0 \preceq \sigma(i)\Big(\hat{x}_i(s) - x_i(s)\Big) \quad \forall s \in [0,1], \ \forall i = 1, \ldots, N.$$

there exists $\hat{k} \in L^1([0,1], E)$ *such that a.e.* $s \in [0,1]$, $\hat{k}(s) \in K(s, \hat{x}(s))$,

$$\|k(s) + ax(s) - \hat{k}(s) - a\hat{x}(s)\| \leq l\|\hat{x} - x\|_0,$$

and

$$0 \preceq \sigma(i)\Big(\hat{k}_i(s) + a\hat{x}_i(s) - k_i(s) - ax_i(s)\Big) \quad \forall i = 1, \ldots, N.$$

(iii) *For every* $i = 1, \ldots, N$*, and every sequence* $\{a_n\}$ *in* E_i *such that* $a_n \to a$ *and* $0 \preceq \sigma(i)(a_{n+1} - a_n)$ *for all* $n \in \mathbb{N}$*, one has* $0 \preceq \sigma(i)(a - a_n)$ *for all* $n \in \mathbb{N}$*.*

Then, (2.3.7) *has a solution.*

Remark 2.5.2. *In the particular case where* $N = 1$*,* $E_1 = \mathbb{R}$ *and* K *is a continuous single-valued map, the previous corollary is due to Nieto and Rodríguez-López [36].*

2.6. Other existence results

It could happen that a map H_i satisfies at the same time a nonincreasing type condition and a nondecreasing type condition with respect to some variables. For instance, for $x, \hat{x}, \tilde{x} \in E$ such that $\tilde{x}_1 \preceq x_1 \preceq \hat{x}_1$ and $x_j = \hat{x}_j = \tilde{x}_j$ for $j \neq 1$, and for $w_i \in H_i(t,s,x)$, there could exist $\hat{w}_i \in H_i(t,s,\hat{x})$, $\tilde{w}_i \in H_i(t,s,\tilde{x})$ such that $w_i \preceq \hat{w}_i$ and $w_i \preceq \tilde{w}_i$. If such H_i is single-valued, that would imply that H_i is constant with respect to x_1. However, in the multivalued case, such H_i does not need to be constant with respect to x_1. In this section, we consider the system (2.1.1) in which some of the maps H_i satisfy this type of property. In some sense, we combine assumptions used in Sections 3 and 5.

In order to simplify the notation, we write $E = E_* \times E_{**}$ with

$$E_* = \prod_{i=1}^{N_*} E_i \quad \text{and} \quad E_{**} = \prod_{i=1}^{N_{**}} E_{i+N_*},$$

where $N = N_* + N_{**}$. We write $(x, y) \in E$ with $x \in E_*$ and $y \in E_{**}$.

Theorem 2.6.1. *Let* $H : [0,1] \times [0,1] \times E \to E$ *be a multivalued map with nonempty values. Assume the following conditions hold:*

(i) *There exist* $(x_0, y_0) \in C([0,1], E)$, $(v_0, w_0) \in \mathcal{H}(x_0, y_0)$, $\sigma_* : \{1, \ldots, N_*\} \to \{1, -1\}$ *and* $\sigma_{**} : \{1, \ldots, N_{**}\} \to \{1, -1\}$ *such that*

$$0 \preceq \sigma_*(i) \left(\int_0^1 v_{0,i}(t,s)\, ds - x_{0,i}(t) \right) \quad \forall t \in [0,1],\ \forall i = 1, \ldots, N_*;$$

$$0 \preceq \sigma_{**}(i) \left(\int_0^1 w_{0,i}(t,s)\, ds - y_{0,i}(t) \right) \quad \forall t \in [0,1],\ \forall i = 1, \ldots, N_{**}.$$

(ii) *There exists* $\phi : [0,1] \times [0,1] \to [0, \infty[$ *such that*

$$\phi(t, \cdot) \in L^1([0,1]) \ \forall t \in [0,1] \quad \textit{and} \quad \sup_{t \in [0,1]} \|\phi\|_1 < 1,$$

and, for $j = 0, 1, \ldots, N_{**}$, *there is a map* $\sigma_j : \{1, \ldots, N_{**}\} \to \{1, -1\}$ *such that, for every* $(x, y) \in C([0,1], E)$, $(v, w) \in \mathcal{H}(x, y)$, *one has*

(a) *for every* $\hat{x} \in C([0,1], E_*)$ *such that*

$$0 \preceq \sigma_*(i) \Big(\hat{x}_i(s) - x_i(s) \Big) \quad \forall s \in [0,1],\ \forall i = 1, \ldots, N_*,$$

there exists $(\hat{v}, \hat{w}) \in \mathcal{H}(\hat{x}, y)$ *such that, a.e.* $s \in [0,1]$ *and all* $t \in [0,1]$,

$$\|(v(t,s), w(t,s)) - (\hat{v}(t,s), \hat{w}(t,s))\| \le \phi(t,s) \|x - \hat{x}\|_0,$$

and

$$0 \preceq \sigma_*(i)(\hat{v}_i(t,s) - v_i(t,s)) \quad \forall i = 1, \ldots, N_*,$$
$$0 \preceq \sigma_0(i)(\hat{w}_i(t,s) - w_i(t,s)) \quad \forall i = 1, \ldots, N_{**};$$

(b) *for* $j = 1, \ldots, N_{**}$, *and every* $\hat{y} \in C([0,1], E_{**})$ *such that* $y_i = \hat{y}_i$ *for* $i \ne j$ *and* $y_j(s) \preceq \hat{y}_j(s)$ *for all* $s \in [0,1]$ *(resp.* $\hat{y}_j(s) \preceq y_j(s)$ *for all* $s \in [0,1]$*), there exists* $(\hat{v}, \hat{w}) \in \mathcal{H}(x, \hat{y})$ *such that, a.e.* $s \in [0,1]$ *and all* $t \in [0,1]$,

$$\|(v(t,s), w(t,s)) - (\hat{v}(t,s), \hat{w}(t,s))\| \le \phi(t,s) \|y - \hat{y}\|_0,$$

and

$$0 \preceq \sigma_*(i)(\hat{v}_i(t,s) - v_i(t,s)) \quad \forall i = 1, \ldots, N_*,$$
$$0 \preceq \sigma_j(i)(\hat{w}_i(t,s) - w_i(t,s)) \quad \forall i = 1, \ldots, N_{**},$$
$$\textit{(resp. } 0 \preceq \sigma_j(i)(w_i(t,s) - \hat{w}_i(t,s))) \quad \forall i = 1, \ldots, N_{**}).$$

(iii) *For every* $i = 1, \ldots, N_*$, *and every sequence* $\{a_n\}$ *in* E_i *such that* $a_n \to a$ *and* $0 \preceq \sigma_*(i)(a_{n+1} - a_n)$ *for all* $n \in \mathbb{N}$, *one has* $0 \preceq \sigma_*(i)(a - a_n)$ *for all* $n \in \mathbb{N}$.

(iv) *For every* $(x, y) \in C([0,1], E)$, $y_n \in C([0,1], E_{**})$, $(v, w) \in \mathcal{E}$ *and* $(v_n, w_n) \in \mathcal{H}(x, y_n)$, *one has* $(v, w) \in \mathcal{H}(x, y)$ *if*

(a) $y_n \to y$ *and* $y_n(t) = \int_0^1 w_{n-1}(t,s)\,ds$ *for all* $t \in [0,1]$ *and all* $n \in \mathbb{N}$*;*

(b) $(v_n(t,s), w_n(t,s)) \to (v(t,s), w(t,s))$ *and*

$$\|(v_n(t,s), w_n(t,s))\| \le M\phi(t,s) + \|(v_0(t,s), w_0(t,s))\|$$
$$a.e.\ \ s \in [0,1], \forall t \in [0,1], \forall n \in \mathbb{N}, \textit{and for some } M \ge 0.$$

Then, (2.1.1) *has a solution.*

Proof. We consider on $C([0,1],E)$ the graph G^* with $V(G^*) = C([0,1],E)$ and

(E(G*)) one has $\big((x,y),(\hat{x},\hat{y})\big) \in E(G^*)$ if and only if one of the following conditions holds:

(i) $y = \hat{y}$ and $0 \preceq \sigma_*(i)\big(\hat{x}_i(s) - x_i(s)\big)$ for all $s \in [0,1]$ and all $i = 1,\dots,N_*$;

(ii) $x = \hat{x}$ and there exists $j \in \{1,\dots,N_{**}\}$ such that, for all $s \in [0,1]$, $y_j(s) \preceq \hat{y}_j(s)$ and $y_i(s) = \hat{y}_i(s)$ for all $i \neq j$;

(iii) $x = \hat{x}$ and there exists $j \in \{1,\dots,N_{**}\}$ such that, for all $s \in [0,1]$, $\hat{y}_j(s) \preceq y_j(s)$ and $y_i(s) = \hat{y}_i(s)$ for all $i \neq j$.

We consider the following properties with $m = N_{**} + 1$:

(P_1) For $\big((x,y),(\hat{x},\hat{y})\big) \in E(G^*)$, we say that $\big((u,\mu),(\hat{u},\hat{\mu})\big) \in \mathcal{P}_1^*\big((x,y),(\hat{x},\hat{y})\big)$ if for all $(v,w) \in \mathcal{H}(x,y)$ such that

$$\big(u(t),\mu(t)\big) = \left(\int_0^1 v(t,s)\,ds, \int_0^1 w(t,s)\,ds\right) \quad \forall t \in [0,1],$$

there exists $(\hat{v},\hat{w}) \in \mathcal{H}(\hat{x},\hat{y})$ such that

(a) one has

$$\big(\hat{u}(t),\hat{\mu}(t)\big) = \left(\int_0^1 \hat{v}(t,s)\,ds, \int_0^1 \hat{w}(t,s)\,ds\right) \quad \forall t \in [0,1];$$

(b) $\Big\|(v(t,s),w(t,s)) - (\hat{v}(t,s),\hat{w}(t,s))\Big\| \le \phi(t,s)\Big\|(x,y)-(\hat{x},\hat{y})\Big\|_0$ a.e. $s \in [0,1]$ and all $t \in [0,1]$;

(c) $0 \preceq \sigma_*(i)(\hat{v}_i(t,s) - v_i(t,s))$ a.e. $s \in [0,1]$, all $t \in [0,1]$, and all $i = 1,\dots,N_*$.

(P_m) For $(\hat{x},\hat{y}) \in [(x,y)]_G^m$, we say that $\big((u,\mu),(\hat{u},\hat{\mu})\big) \in \mathcal{P}_m\big((x,y),(\hat{x},\hat{y})\big)$ if there exist $\big((x^k,y^k)\big)_{k=0}^m$ and $\big((u^k,\mu^k)\big)_{k=0}^m$ m-directed paths from (x,y) to $(\hat{x},\hat{y})$ and from (u,μ) to $(\hat{u},\hat{\mu})$ respectively such that $\big((u^{k-1},\mu^{k-1}),(u^k,\mu^k)\big) \in \mathcal{P}_1^*\big((x^{k-1},y^{k-1}),(x^k,y^k)\big)$ for $k = 1,\dots,m$. In particular, for all $(v,w) \in \mathcal{H}(x,y)$ such that

$$\big(u(t),\mu(t)\big) = \left(\int_0^1 v(t,s)\,ds, \int_0^1 w(t,s)\,ds\right),$$

there exists $(\hat{v},\hat{w}) \in \mathcal{H}(\hat{x},\hat{y})$ such that, a.e. $s \in [0,1]$ and all $t \in [0,1]$,

$$\Big(\hat{u}(t),\hat{\mu}(t)\Big) = \left(\int_0^1 \hat{v}(t,s)\,ds, \int_0^1 \hat{w}(t,s)\,ds\right),$$

$$\Big\|(v(t,s),w(t,s)) - (\hat{v}(t,s),\hat{w}(t,s))\Big\| \le \phi(t,s)p_m\Big((x,y),(\hat{x},\hat{y})\Big),$$

and

$$0 \preceq \sigma_*(i)(\hat{v}_i(t,s) - v_i(t,s)) \quad \forall i = 1,\dots,N_*.$$

(P_{m^n}) For $n \ge 2$ and $(\hat{x},\hat{y}) \in [(x,y)]_G^{m^n}$, the property $\mathcal{P}^*_{m^n}\Big((x,y),(\hat{x},\hat{y})\Big)$ is defined inductively. Hence, for $\Big((u,\mu),(\hat{u},\hat{\mu})\Big) \in \mathcal{P}_{m^n}\Big((x,y),(\hat{x},\hat{y})\Big)$, one has $(\hat{u},\hat{\mu}) \in [(x,y)]_G^{m^{n+1}}$ and for all $(v,w) \in \mathcal{H}(x,y)$ such that

$$\Big(u(t),\mu(t)\Big) = \left(\int_0^1 v(t,s)\,ds, \int_0^1 w(t,s)\,ds\right),$$

there exists $(\hat{v},\hat{w}) \in \mathcal{H}(\hat{x},\hat{y})$ such that, a.e. $s \in [0,1]$ and all $t \in [0,1]$,

$$\Big(\hat{u}(t),\hat{\mu(t)}\Big) = \left(\int_0^1 \hat{v}(t,s)\,ds, \int_0^1 \hat{w}(t,s)\,ds\right),$$

$$\Big\|(v(t,s),w(t,s)) - (\hat{v}(t,s),\hat{w}(t,s))\Big\| \le \phi(t,s)p_{m^n}\Big((x,y),(\hat{x},\hat{y})\Big),$$

and

$$0 \preceq \sigma_*(i)(\hat{v}_i(t,s) - v_i(t,s)) \quad \forall i = 1,\dots,N_*.$$

It follows from (i) and (ii) that Assumptions (i) and (ii) of Theorem 2.2.4 are satisfied.

Finally, (iii) and (iv) imply that Condition (iii) of Theorem 2.2.4 is satisfied. Indeed, Let $\{(x_n,y_n)\}$ be such that $(x_n,y_n) \in F(x_{n-1},y_{n-1}) \cap [(x_{n-1},y_{n-1})]_G^{m^n}$ and $\Big((x_n,y_n),(x_{n+1},y_{n+1})\Big) \in \mathcal{P}_{m^n}\Big((x_{n-1},y_{n-1}),(x_n,y_n)\Big)$ for all $n \in \mathbb{N}$ and

$$\sum_{n=1}^{\infty} p_{m^{n+1}}\Big((x_n,y_n),(x_{n+1},y_{n+1})\Big) < \infty.$$

So, there exist $(x,y) \in C([0,1],E)$ and (v,w) such that $\|(x_n,y_n)-(x,y)\|_0 \to 0$, and a.e. $s \in [0,1]$ for all $t \in [0,1]$, $\Big\|(v(t,\cdot),w(t,\cdot)) - (v_n(t,\cdot),w_n(t,\cdot))\Big\|_1 \to 0$, $(v_n(t,s),w_n(t,s)) \to (v(t,\cdot),w(t,\cdot))$ and, by (iii), $0 \preceq \sigma_*(i)(x_i(t,s) - x_{n,i}(t,s))$ for all $i = 1,\dots,N_*$.

It follows from (ii), that for every $n \in \mathbb{N}$, there exists $(\hat{v}_n,\hat{w}_n) \in \mathcal{H}(x,y_n)$ such that

$$\Big\|(v_n(t,s),w_n(t,s)) - (\hat{v}_n(t,s),\hat{w}_n(t,s))\Big\| \le \phi(t,s)\|x_n - x\|_0 \quad \text{a.e. } s \in [0,1],\ \forall t \in [0,1].$$

Therefore,

$$(\hat{v}_n(t,s),\hat{w}_n(t,s)) \to (v(t,s),w(t,s)) \quad \text{a.e. } s \in [0,1],\ \forall t \in [0,1].$$

Assumption (iv) implies that $(v,w) \in \mathcal{H}(x,y)$, and hence $(x,y) \in F(x,y)$.

Finally, Theorem 2.2.4 gives the conclusion. $\square$

Chapter 3

APPLICATIONS OF MULTIVALUED CONTRACTIONS ON GRAPHS TO GRAPH-DIRECTED ITERATED FUNCTION SYSTEMS

3.1. Introduction

Based on the work of Hutchinson [28] and popularized by Barnsley [2], the method of iterated function systems (IFS) permits to generate fractals by iterating a collection of transformations $\{T_i : i = 1, \ldots, p\}$. If each T_i is a contraction on a complete metric space M, it was shown in [28] that there exists a unique nonempty compact set $K \subset M$ which is invariant with respect to $\{T_i : i = 1, \ldots, p\}$; that is

$$K = \bigcup_{i=1}^{p} T_i(K).$$

This attractor K is such that, for every compact $A \subset M$,

$$g^n(A) \to K \quad \text{in the sense of the Hausdorff metric,}$$

where

$$g(A) = \bigcup_{i=1}^{p} T_i(A).$$

The existence of K can be deduced from the Banach fixed point theorem.

A fixed point result which is, in some sense, a combination of the Banach contraction principle and the Knaster-Tarski fixed point theorem in a partially ordered set was obtained by Ran and Reurings [44] in 2004. They considered a monotone, order preserving single-valued map f defined on a complete metric space endowed with a partial ordering. They assumed that f satisfies a contraction condition not necessarily for all x and y, but for those such that $x \leq y$.

Subsequently, their result was generalized by many authors, in particular by Nieto, Rodríguez-López, Pouso, Petruşel and Rus [35, 36, 37, 43]. In 2008, Jachymski [29] presented a nice unification of most of the previous results by considering complete metric spaces endowed with a graph G. He introduced the notion of single-valued G-contraction for which he obtained fixed point results.

Using those fixed point results, Gwóźdź-Łukawska and Jachymski [25] developped the Hutchinson-Barnsley theory on complete metric space endowed with a graph G for iterated function systems of single-valued G-contractions.

Different extensions of the concept of single-valued G contractions on complete metric spaces endowed with a graph to multivalued maps were presented by Dinevari and Frigon [10] and by Nicolae, O'Regan, and Petruşel [34]. Those extentions led to generalizations of Jachymski's fixed point results and of the Nadler fixed point theorem for multivalued contractions.

In 1988, Mauldin and Williams [31] introduced the notion of geometric graph-directed construction.

Definition 3.1.1. *A geometric graph-directed construction in $\mathbb{R}^m$ consists of*

(i) a collection of p non-overlapping, compact, nonempty subsets of $\mathbb{R}^m$, $J_1, \ldots, J_p$ with nonempty interior;

(ii) a directed graph $H = (V(H), E(H))$ such that $V(H) = \{1, \ldots, p\}$ is the set of its vertices, and, for each $i \in V(H)$, there exists some edge $(i,j) \in E(H)$;

(iii) for each $(i,j) \in E(H)$, there is a similarity map $T_{i,j} : \mathbb{R}^m \to \mathbb{R}^m$ with similarity ratios $r_{i,j}$ such that

$$\bigcup_{(i,j)\in E(H)} T_{i,j}(J_j) \subset J_i;$$

(iv) for each i, $\{T_{i,j}(J_j) : (i,j) \in E(H)\}$ is a nonoverlapping family of sets;

(v) if $[i_1, \ldots, i_{q-1}, i_q = i_1]$ is a cycle in H, then

$$\prod_{k=1}^{q} r_{i_{k-1},i_k} < 1.$$

They showed that a geometric graph-directed construction has an attractor.

Theorem 3.1.1 (Mauldin and Williams [31])**.** *For a geometric graph-directed construction as above, there exists $K_1, \ldots, K_p$ a unique collection of nonempty compact sets such that*

$$\forall i \in \{1, \ldots, p\},\ K_i \subset J_i \quad \textit{and} \quad K_i = \bigcup_{(i,j)\in E(H)} T_{i,j}(K_j).$$

The set

$$K = \bigcup_{i=1}^{p} K_i$$

is called the attractor *of this geometric graph-directed construction.*

Geometric graph-directed constructions have been studied and generalized by many authors, see [8, 9, 14, 15]. In particular, it was shown in [8] that with an appropriate rescaling, condition (v) can be replaced by

(v)' for each $(i, j) \in H$, $r_{i,j} < 1$.

Also, in some of those generalizations, similarities on $\mathbb{R}^m$ were replaced by contractions on complete metric spaces and the terminology of graph-directed iterated function system was used. Again, the existence of an attractor K was established.

In this paper, we take into account the graph H to obtain more information on the attractor K of a graph-directed iterated function system. To do so, we apply a fixed point result obtained by the authors [10] for multivalued contractions on complete metric spaces endowed with a graph.

The paper is organized as follows. In section 2, we present some notations and we recall some results. In section 3, we consider a space X such that $K \in X$ and on which we define a suitable graph G and a suitable metric. In section 4, we define an appropriate multivalued G-contraction F. In the last three sections, taking into account the maximal connected component of the graph H, we obtain more information on the attractor K from some fixed points of F.

3.2. H-iterated function systems

First of all, we introduce the notion of MW-directed graph and we consider iterated function systems which takes into account the structure of an MW-directed graph.

Definition 3.2.1. *A directed graph $H = (V(H), E(H))$ is called an MW-directed graph if $V(H) = \{1, \ldots, p\}$, H has no parallel edges, and for every $i \in V(H)$, there exists $j \in V(H)$ such that $(i, j) \in E(H)$.*

Definition 3.2.2. *Let $H = (V(H), E(H))$ be an MW-directed graph. A graph-directed iterated function system over the graph H (H-IFS) is a collection of p nonempty, bounded, complete metric spaces, $(X_1, d_1), \ldots, (X_p, d_p)$, and, for each $(i, j) \in E(H)$, a contraction $T_{i,j} : X_j \to X_i$ with constant of contraction $\lambda_{i,j}$. An H-IFS is denoted by $\{T_{i,j}\}_H$.*

Definition 3.2.3. *Let $\{T_{i,j}\}_H$ be an H-IFS. An attractor K of the H-IFS is a collection of nonempty compact sets $K = \{K_i\}_H$ such that $K_i \subset X_i$ and*

$$K_i = \bigcup_{(i,j)\in E(H)} T_{i,j}(K_j) \quad \forall i \in \{1, \ldots, p\}.$$

The Banach contraction principle insures the existence of an attractor of an H-IFS. We present the proof for sake of completeness. For more information on graph-directed iterated function systems, the reader is referred to [14, 31].

Theorem 3.2.1. *An H-IFS, $\{T_{i,j}\}_H$, has a unique attractor K.*

Proof. Consider

$$Y = \left\{ (S_1, \dots, S_p) \subset \prod_{i=1}^{p} X_i : S_i \text{ is a compact nonempty subset of } X_i \right\}$$

endowed with the metric $\rho(S, \widehat{S}) = \max\{D_i(S_i, \widehat{S}_i) : i = 1, \dots, p\}$, where D_i is the Hausdorff metric on X_i, that is

$$D_i(S_i, \widehat{S}_i) = \inf\{\varepsilon > 0 : S_i \subset B(\widehat{S}_i, \varepsilon) \text{ and } \widehat{S}_i \subset B(S_i, \varepsilon)\},$$

where

$$B(S_i, \varepsilon) = \{y \in X_i : \exists x \in S_i \text{ such that } d_i(x, y) < \varepsilon\}.$$

Let us define $f : Y \to Y$ by

$$f_i(S) = \bigcup_{(i,j) \in E(H)} T_{i,j}(S_j).$$

Using the fact that every $T_{i,j}$ is a contraction, one verifies that f is a contraction with constant of contraction $\theta = \max\{\lambda_{i,j} : (i,j) \in E(H)\}$. The Banach contraction principle insures the existence of $K \in Y$ a unique fixed point of f. Thus, K is the unique attractor of $\{T_{i,j}\}_H$. $\square$

More information on K will be obtained by applying a fixed point result for multivalued contractions on complete metric spaces endowed with a graph. We recall the notion of G-contraction introduced in [10].

For (X, d) a complete metric space, we consider $G = (V(G), E(G))$ a directed graph such that $X = V(G)$, the diagonal in $X \times X$ is contained in $E(G)$, and G has no parallel edges.

Definition 3.2.4. *Let $F : X \to X$ be a multivalued map with nonempty values. We say that F is a G-contraction if there exists $\alpha \in]0, 1[$ such that*

(C_G) for all $(x, y) \in E(G)$ and all $u \in F(x)$, there exists $v \in F(y)$ such that $(u, v) \in E(G)$ and $d(u, v) \leq \alpha d(x, y)$.

We consider suitable trajectories in X.

Definition 3.2.5. *Let $F : X \to X$ be a multivalued mapping and $x_0 \in X$. We say that a sequence $\{x_n\}$ is a G_1-Picard trajectory from x_0 if $x_n \in F(x_{n-1})$ and $(x_{n-1}, x_n) \in E(G)$ for all $n \in \mathbb{N}$. The set of all such G_1-Picard trajectories from x_0 is denoted by $T_1(F, G, x_0)$.*

The reader is referred to [10] for the proof of the following fixed point result for multivalued G-contractions.

Theorem 3.2.2. *Let $F : X \to X$ be a multivalued G-contraction such that there exists $(x_0, x_1) \in E(G)$ such that $x_1 \in F(x_0)$. In addition, assume that one of the following conditions holds:*

(i) *F is G_1*-Picard continuous from *x_0, i.e. the limit of any convergent sequence $\{x_n\} \in T_1(F, G, x_0)$ is a fixed point of F.*

(ii) *F has closed values and, for every $\{x_n\}$ in $T_1(F, G, x_0)$ converging to some $x \in X$, there exists a subsequence $\{n_k\}$ such that $(x_{n_k}, x) \in E(G)$ for all $k \in \mathbb{N}$.*

Then, there exists a G_1-Picard trajectory from x_0, $\{x_n\}$, converging to x a fixed point of F. Moreover, every converging G_1-Picard trajectory from x_0 converges to a fixed point of F.

In what follows, we consider H a MW-directed graph. We will use the following definitions and notations.

A *path from i to j in H* is denoted $[i_k]_0^N = [i_0, \ldots, i_N]$ where $i = i_0$, $j = i_N$ and $(i_{k-1}, i_k) \in E(H)$ for every $k = 1, \ldots, N$.

We say that a subgraph $C = (V(C), E(C))$ of H is *connected* if for every $i, j \in V(C)$ there exists a path from i to j in C. A *connected component* of H is a maximal connected subgraph of H. We denote

$$C(H) = \{C \text{ connected component of } H\}.$$

It follows from the definition of MW-directed graph that

$$\emptyset \neq C(H) = \{C_\alpha : \alpha \in \Lambda\}, \quad \text{where } \Lambda \text{ has finite cardinality.}$$

We can define a partial order on $C(H)$ as follows:

$$C_\alpha \preceq C_\beta \quad \iff \quad \exists [i_k]_0^N \text{ a path in } H \text{ such that } i_0 \in C_\alpha \text{ and } i_N \in C_\beta.$$

We write $C_\alpha \prec C_\beta$ to mean $C_\alpha \preceq C_\beta$ and $C_\alpha \neq C_\beta$. We say that C_α and C_β are *incomparable* if $C_\alpha \npreceq C_\beta$ and $C_\beta \npreceq C_\alpha$.

We denote the set of vertices from which there is a path in H reaching $i \in V(H)$ by

$$[i]_\leftarrow = \{j \in V(H) : \text{ there is a path from } j \text{ to } i \text{ in } H\}. \tag{3.2.1}$$

Similarly, for $C \in C(H)$, we denote the set of vertices from which there is a path in H reaching $V(C)$ by

$$[C]_\leftarrow = \bigcup_{i \in V(C)} [i]_\leftarrow. \tag{3.2.2}$$

3.3. A suitable metric space endowed with a directed graph

Let H be a MW-directed graph with $V(H) = \{1, \dots, p\}$. For $i \in V(H)$, let (X_i, d_i) be a bounded complete metric space.

In this section, using H and the spaces X_i, we define a complete metric space endowed with a suitable directed graph. Let us recall that

$$C(H) = \{C \text{ connected component of } H\}.$$

We consider the space X of sets $A = (A_1, \dots, A_p)$ satisfying the following properties:

(Xi) $A_i \subset X_i$ is compact for every $i = 1, \dots, p$;

(Xii) if $A_i \neq \emptyset$ for some $i \in V(C)$ and $C \in C(H)$, then $A_j \neq \emptyset$ for all $j \in V(C)$;

(Xiii) there exists $C \in C(H)$ and $i \in V(C)$ such that $A_i \neq \emptyset$.

It is important to point out that for $A = (A_1, \dots, A_p) \in X$, some A_i can be empty.

We endow X with the metric

$$d(A, B) = \max_{i \in \{1, \dots, p\}} \overline{D}_i(A_i, B_i), \tag{3.3.1}$$

where

$$\overline{D}_i(A_i, B_i) = \begin{cases} D_i(A_i, B_i), & \text{if } A_i \neq \emptyset, B_i \neq \emptyset, \\ 0, & \text{if } A_i = \emptyset = B_i, \\ R_i, & \text{otherwise,} \end{cases} \tag{3.3.2}$$

where D_i the Hausdorff metric in X_i and $R_i > R$ is a constant which will be fixed later, with

$$R = \max\{\operatorname{diam}(X_i) : i = 1, \dots, p\}.$$

It is clear that (X, d) is a complete metric space.

Taking into account the graph H, we want to endow X with a directed graph. To do so, we distinguish vertices of H which are in a connected component from the others. We set

$$V^c = \bigcup_{C \in C(H)} V(C) \tag{3.3.3}$$

and

$$V^e = V(H) \backslash V^c. \tag{3.3.4}$$

We define the graph G as follows: $V(G) = X$, and for $A, B \in X$, $(A, B) \in E(G)$ if and only if

(G) for every $i \in \{1, \dots, p\}$, one of the following properties holds:

(i) $A_i = B_i = \emptyset$, or $A_i \neq \emptyset$ and $B_i \neq \emptyset$;

(ii) $A_i = \emptyset$, $B_i \neq \emptyset$, and one of the following statements is true:

(a) $i \in V^e$ and there exists $j \in V(H)$ such that $(i, j) \in E(H)$ and $A_j \neq \emptyset$;

(b) $i \in V(C)$ for some $C \in C(H)$ and there exist $k \in V(C)$ and $j \in V(H)$ such that $(k, j) \in E(H)$ and $A_j \neq \emptyset$;

(iii) $A_i \neq \emptyset$, $B_i = \emptyset$, $i \in V^e$, and one of the following properties is satisfied:

(a) there is no $j \in V(H)$ such that $(j, i) \in E(H)$;

(b) for every $j \in V(H)$ such that $(j, i) \in E(H)$, one has $B_j \neq \emptyset$.

Example 3.3.1. *Let H be the MW-graph of Figure 3.1. We consider X the*

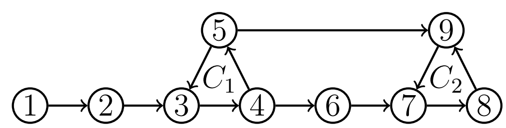

Fig. 3.1. An MW-graph H.

associated metric space satisfying (Xi)–(Xiii) endowed with the graph G satisfying the condition (G). Let A_i^k be nonempty compact subsets of X_i for all $i \in \{1, \dots, 9\}$ and $k \in \{1, \dots, 7\}$. We consider the following elements of X:

$$A^1 = \Big(\emptyset, \emptyset, A_3^1, A_4^1, A_5^1, \emptyset, \emptyset, \emptyset, \emptyset\Big), \qquad A^2 = \Big(\emptyset, \emptyset, A_3^2, A_4^2, A_5^2, A_6^2, \emptyset, \emptyset, \emptyset\Big),$$

$$A^3 = \Big(\emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, A_7^3, A_8^3, A_9^3\Big), \qquad A^4 = \Big(\emptyset, \emptyset, \emptyset, \emptyset, \emptyset, A_6^4, A_7^4, A_8^4, A_9^4\Big),$$

$$A^5 = \Big(\emptyset, \emptyset, A_3^5, A_4^5, A_5^5, \emptyset, A_7^5, A_8^5, A_9^5\Big), \qquad A^6 = \Big(A_1^6, \emptyset, A_3^6, A_4^6, A_5^6, \emptyset, \emptyset, \emptyset, \emptyset\Big)$$

$$A^7 = \Big(\emptyset, A_2^7, A_3^7, A_4^7, A_5^7, \emptyset, \emptyset, \emptyset, \emptyset\Big).$$

Here is the list of all edges of G between them:

$$\Big\{(A^1, A^7), (A^2, A^1), (A^2, A^7), (A^3, A^4), (A^3, A^4), (A^4, A^5),$$
$$(A^6, A^1), (A^6, A^7), (A^7, A^6)\Big\} \subset E(G).$$

Now, we want to fix R_i in (3.3.2) in such a way that we will be able to define a suitable multivalued G-contraction on X in the next section. To this aim, we decompose $V(H)$ in appropriate subsets V_μ with $\mu \in I$ a totally ordered set.

Lemma 3.3.1. *Let H be a MW-directed graph. Then there exist I a totally ordered set, $\{V_\mu : \mu \in I\}$ a family of non empty disjoint subsets, and, for every $i \in \{1, \dots, p\}$, there exists $R_i > R$ such that*

(1) $V(H) = \cup_{\mu \in I} V_\mu$;

(2) if $V(C) \cap V_\mu \neq \emptyset$ for some $\mu \in I$ and some $C \in C(H)$, then $V(C) \subset V_\mu$;

(3) if $\mu < \nu$ in I, for all $i \in V_\mu$, and $j \in V_\nu$, then $j \notin [i]_\leftarrow$;

(4) for every $\mu \in I$, one has $R_i = R_j$ for every $i, j \in V_\mu$;
(5) for every $\mu < \nu \in I$, one has $R_i < R_j$ for every $i \in V_\mu$, $j \in V_\nu$.

Proof. We want to separate vertices of H in suitable subsets. Let us recall that some vertices are in a connected component, and some others are not:

$$V(H) = V^c \cup V^e,$$

where V^c and V^e are defined in (3.3.3) and (3.3.4) respectively.

First of all, we examine vertices in V^c. Let

$$L = \max\Big\{n \in \mathbb{N} : \text{ there exists a chain } C_{\alpha_1} \prec \cdots \prec C_{\alpha_n} \text{ in } C(H)\Big\}. \tag{3.3.5}$$

We denote

$$\begin{aligned}
C(H)_1 &= \Big\{C \in C(H) : \nexists \widehat{C} \in C(H) \text{ such that } \widehat{C} \prec C\Big\},\\
C(H)_2 &= \Big\{C \in C(H)\backslash C(H)_1 : \nexists \widehat{C} \in C(H)\backslash C(H)_1 \text{ such that } \widehat{C} \prec C\Big\},\\
&\ \ \vdots\\
C(H)_L &= \left\{C \in C(H)\backslash \bigcup_{k=1}^{L-1} C(H)_k : \nexists \widehat{C} \in C(H)\backslash \bigcup_{k=1}^{L-1} C(H)_k \text{ such that } \widehat{C} \prec C\right\}.
\end{aligned}$$

We define

$$V_{k,0} = \bigcup_{C \in C(H)_k} V(C) \quad \text{for } k = 1, \dots, L.$$

Observe that

$$V^c = \bigcup_{k=1}^{L} V_{k,0} \quad \text{and} \quad V_{k,0} \cap V_{j,0} = \emptyset \quad \text{if } k \neq j.$$

Now, we separate vertices in V^e in suitable subsets. We first separate them in two sets: those which can be reached by a path starting from a vertex in a connected component, and those which cannot. This last set is denoted

$$V^0 = \{j \in V^e : V^c \cap [j]_{\leftarrow} = \emptyset\}. \tag{3.3.6}$$

If $V^0 \neq \emptyset$, let

$$N_0 = \max\Big\{n : \text{ there is a path } [i_k]_1^n \text{ such that } i_k \in V^0 \text{ for every } k = 1, \dots, N_0\Big\}.$$

We define

$$\begin{aligned}
V_{0,1} &= \Big\{i \in V^0 : \nexists j \in V^0 \text{ such that } (j, i) \in E(H)\Big\},\\
V_{0,2} &= \Big\{i \in V^0 \backslash V_{0,1} : \nexists j \in V^0 \backslash V_{0,1} \text{ such that } (j, i) \in E(H)\Big\},
\end{aligned}$$

$$\vdots$$

$$V_{0,N_0} = \left\{ i \in V^0 \backslash \bigcup_{k=1}^{N_0-1} V_{0,k} : \nexists j \in V^0 \backslash \bigcup_{k=1}^{N_0-1} V_{0,k} \text{ such that } (j,i) \in E(H) \right\}.$$

Observe that

$$V^0 = \bigcup_{k=1}^{N_0} V_{0,k} \quad \text{and} \quad V_{0,k} \cap V_{0,j} = \emptyset \quad \text{if } k \neq j.$$

If $V^e \backslash V^0 \neq \emptyset$, it follows from Definition 3.2.1 that for every $j \in V^e \backslash V^0$, there exist $C_\alpha, C_\beta \in C(H)$ such that

$$C_\alpha \prec C_\beta, \quad V(C_\alpha) \subset [j]_{\leftarrow} \quad \text{and} \quad j \in [C_\beta]_{\leftarrow}.$$

In other words, j is on a path from from C_α to C_β. Hence $L > 1$, where L is defined in (3.3.5).

If $L \geq 2$, we first examine vertices on a path from some $i \in V_{1,0}$ to some $j \in V_{2,0}$. Let

$$N_1 = \max \Big\{ n : \text{ there is a path } [i_k]_0^{1+n} \text{ such that } i_0 \in V_{1,0},\ i_{1+n} \in V_{2,0} \\ \text{and } i_k \in V^e \text{ for all } k = 1, \ldots, n \Big\}.$$

If $N_1 \geq 1$, we define

$$V_{1,1} = \Big\{ i \in V^e : \exists [i_k]_0^{1+N_1} \text{ with } i = i_1,\ i_0 \in V_{1,0},\ i_{1+N_1} \in V_{2,0}, \\ i_k \in V^e \text{ for } k = 1, \ldots, N_1 \Big\}.$$

If $N_1 \geq 2$, we define

$$V_{1,2} = \Big\{ i \in V^e \backslash V_{1,1} : \exists [i_k]_1^{1+N_1} \text{ with } i = i_2,\ i_1 \in V_{1,0} \cup V_{1,1},\ i_{1+N_1} \in V_{2,0}, \\ i_k \in V^e \backslash V_{1,1} \text{ for } k = 2, \ldots, N_1 \Big\}.$$

We define inductively $V_{1,1}, \ldots, V_{1,N_1}$.

We denote the set of vertices on a path from $V_{1,0}$ to $V_{2,0}$ by

$$V^1 = V_{1,0} \cup V_{2,0} \cup \bigcup_{k=1}^{N_1} V_{1,k}.$$

If $L \geq 3$, we examine vertices on a path from some $i \in V^1$ to some $j \in V_{3,0}$. Let

$$N_2 = \max \Big\{ n : \text{ there is a path } [i_k]_0^{1+n} \text{ such that } i_0 \in V^1,\ i_{1+n} \in V_{3,0} \\ \text{and } i_k \in V^e \backslash V^1 \text{ for all } k = 1, \ldots, n \Big\}.$$

If $N_2 \geq 1$, we define

$$V_{2,1} = \Big\{ i \in V^e \backslash V^1 : \exists [i_k]_0^{1+N_2} \text{ with } i = i_1,\ i_0 \in V^1,\ i_{1+N_2} \in V_{3,0},$$
$$i_k \in V^e \backslash V^1 \text{ for } k = 1, \dots, N_2 \Big\}.$$

If $N_2 \geq 2$, we define

$$V_{2,2} = \Big\{ i \in V^e \backslash (V^1 \cup V_{2,1}) : \exists [i_k]_1^{1+N_2} \text{ with } i = i_2,\ i_1 \in V^1 \cup V_{2,1},$$
$$i_{1+N_2} \in V_{3,0}, \text{ and } i_k \in V^e \backslash (V^1 \cup V_{2,1}) \text{ for } k = 2, \dots, N_2 \Big\}.$$

Similarly, we define $V_{2,j}$ for $j \leq N_2$.

So, inductively, we define the following subsets of $V^e \backslash V^0$:

$$V_{1,1}, \dots, V_{1,N_1}, V_{2,1}, \dots, V_{2,N_2}, \dots, V_{L-1,1}, \dots, V_{L-1,N_{L-1}}.$$

Each vertex in one of those sets is on a path from one connected component to an other.

We have decomposed $V(H)$ in a collection of disjoint sets:

$$V_{0,1}, \dots, V_{0,N_0}, V_{1,0}, V_{1,1}, \dots, V_{1,N_1}, V_{2,0}, \dots, V_{L-1,N_{L-1}}, V_{L,0}.$$

We denote

$$I = \Big\{ (k,0) : 1 \leq k \leq L \Big\} \cup \Big\{ (k,l) : 0 \leq k \leq L-1, 1 \leq l \leq N_k \Big\}.$$

We endow I with the order

$$(k_1, l_1) \leq (k_2, l_2) \quad \iff \quad k_1 < k_2, \text{ or } k_1 = k_2, l_1 \leq l_2.$$

By construction,

$$V(H) = \bigcup_{\mu \in I} V_\mu \quad \text{and} \quad V_\mu \cap V_\nu = \emptyset \quad \text{if } \mu \neq \nu.$$

Also, for every $C \in C(H)$, there exists $\mu \in I$ such that $V(C) \subset V_\mu$. Moreover, for $\mu, \nu \in I$ such that $\mu < \nu$, one has $j \notin [i]_\leftarrow$ for every $i \in V_\mu$, and $j \in V_\nu$.

Finally, we choose $\sigma : I \to \,]1, \infty[$ a strictly increasing map. We define

$$R_i = \sigma(\mu) R \quad \text{for } i \in \{1, \dots, p\} \cap V_\mu \text{ and } \mu \in I.$$

By construction, statements (4) and (5) are satisfied. $\square$

Remark 3.3.1. *Let $(A, B) \in E(G)$. From the definition of the graph G and Lemma 3.3.1, we can make the following observations:*

(1) If for some $i \in V(H)$, (G)(ii) holds with some $j \in V(H)$ such that $A_j \neq \emptyset$. Let $\mu_i, \mu_j \in I$ be such that $i \in V_{\mu_i}$ and $j \in V_{\mu_j}$. Then, $\mu_i < \mu_j$.

(2) If for some $i \in V(H)$, (G)(iii)(b) holds, let $\mu_i \in I$ be such that $i \in V_{\mu_i}$. Then, for all $j \in V(H)$ such that $(j,i) \in E(H)$, there is $\mu_j \in I$ such that $j \in V_{\mu_j}$ and one has $\mu_j < \mu_i$.

Example 3.3.2. *We consider H the MW-graph of Figure 3.2 for which we describe the collection of subsets V_μ constructed as in the proof of Lemma 3.3.1. In*

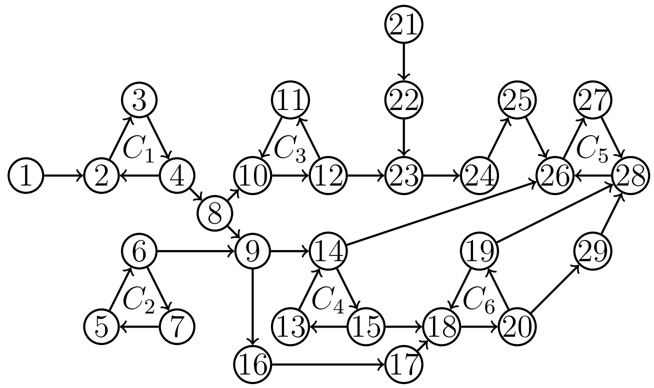

FIG. 3.2. An MW-graph H with $C(H) = \{C_1, \ldots, C_6\}$.

this graph, $C(H) = \{C_1, \ldots, C_6\}$,

$$V^c = \{2,3,4,5,6,7,10,11,12,13,14,15,18,19,20,26,27,28\},$$
$$V^e = \{1,8,9,16,17,21,22,23,24,25,29\}.$$

Since $C_2 \prec C_4 \prec C_6 \prec C_5$, one has $L = 4$, and

$$V_{1,0} = \{2,3,4,5,6,7\},$$
$$V_{2,0} = \{10,11,12,13,14,15\},$$
$$V_{3,0} = \{18,19,20\},$$
$$V_{4,0} = \{26,27,28\}.$$

By considering the paths from $V_{1,0}$ to $V_{2,0}$, one sees that $N_1 = 2$, and

$$V_{1,1} = \{8\}, \quad V_{1,2} = \{9\}.$$

By considering the paths from $V_{1,0} \cup V_{1,1} \cup V_{1,2} \cup V_{2,0}$ to $V_{3,0}$, one sees that $N_2 = 2$, and,

$$V_{2,1} = \{16\}, \quad V_{2,2} = \{17\}.$$

Similarly, one has $N_3 = 3$, and

$$V_{3,1} = \{23\}, \quad V_{3,2} = \{24\}, \quad V_{3,3} = \{25,29\}.$$

So, the vertices which are not in one of the previous sets are in $V^0 = \{1, 21, 22\}$*. Similarly,* $N_0 = 2$*, and*

$$V_{0,1} = \{1, 21\}, \quad V_{0,2} = \{22\}.$$

So, I *is the totally ordered set*

$$(0,1) < (0,2) < (1,0) < (1,1) < (1,2) < (2,0) < (2,1) < (2,2) \\ < (3,0) < (3,1) < (3,2) < (3,3) < (4,0),$$

and

$$V(H) = \{1, \dots, 29\} = \bigcup_{\mu \in I} V_\mu.$$

3.4. A G-contraction

In this section, we consider a graph-directed iterated function system over the graph H, $\{T_{i,j}\}_H$. We will define an appropriate multivalued G-contraction on X, where G and X are respectively the graph and the metric space endowed with this graph and defined in the previous section. This G-contraction will be used to get more information on the attractor of this H-IFS.

Let $A \in X$. For each j such that $A_j \neq \emptyset$, $T_{i,j}(A_j) \neq \emptyset$ for all i such that $(i,j) \in E(H)$. So, it is important to distinguish all those edges. To this aim, we introduce the following notations.

Let V^e be the subset of vertices in $V(H)$ which are not in connected components of H and defined in (3.3.4). So, for $i \in V^e$, we denote

$$E_i(A) = \{(i,j) \in E(H) : A_j \neq \emptyset\}. \tag{3.4.1}$$

For $\emptyset \neq P \subset E_i(A)$, we define

$$U_i^e(A, P) = \bigcup_{(i,j) \in P} T_{i,j}(A_j). \tag{3.4.2}$$

Let V^c be the subset of vertices in $V(H)$ which are in connected components of H and defined in (3.3.3). So, for $i \in V^c$, there exists $C \in C(H)$ such that $i \in V(C)$. We consider the set of edges from a vertex of C to a vertex outside of C for which the component of A is nonempty:

$$E_C(A) = \{(k,j) \in E(H) : k \in V(C), j \notin V(C), A_j \neq \emptyset\}. \tag{3.4.3}$$

For $k \in V(C)$, we denote

$$\{i \xrightarrow{C} k\} = \{[i_k]_0^N \text{ which is a path in } C \text{ from } i = i_0 \text{ to } k = i_N \\ \text{and containing no cycle}\}, \tag{3.4.4}$$

and we define $T_{i\to k} : X_k \to X_i$ by

$$T_{i\to k}(x) = \bigcup_{[i_k]_0^N \in \{i \xrightarrow{C} k\}} T_{i_0,i_1} \circ \cdots \circ T_{i_{N-1},i_N}(x). \tag{3.4.5}$$

We define

$$U_i^c(A,P) = \begin{cases} \emptyset, & \text{if } P = \emptyset, \\ \displaystyle\bigcup_{(k,j)\in P} T_{i\to k} \circ T_{k,j}(A_j), & \text{if } \emptyset \neq P \subset E_C(A). \end{cases} \tag{3.4.6}$$

We also define

$$W_i(A) = \begin{cases} \emptyset, & \text{if } A_i = \emptyset, \\ \displaystyle\bigcup_{(i,j)\in E(C)} T_{i,j}(A_j), & \text{if } A_i \neq \emptyset, \end{cases} \tag{3.4.7}$$

where $E(C) = \{(k,j) \in E(H) : k, j \in V(C)\}$.

We have all the ingredients to define the multivalued map $F : X \to X$. For $A \in X$,

$$U = (U_1, \dots, U_p) \in F(A) \quad \iff \quad U_i \in F_i(A), \tag{3.4.8}$$

where $F_i(A)$ is defined as follows:

For $i \in V^e$,

$$F_i(A) = \begin{cases} \emptyset, & \text{if } E_i(A) = \emptyset, \\ \{U_i^e(A,P) : \emptyset \neq P \subset E_i(A)\}, & \text{if } E_i(A) \neq \emptyset. \end{cases} \tag{3.4.9}$$

For $i \in V(C)$ for some $C \in C(H)$,

$$F_i(A) = \begin{cases} \emptyset, & \text{if } A_i = \emptyset \text{ and } E_C(A) = \emptyset, \\ \{U_i^c(A,P) : \emptyset \neq P \subset E_C(A)\}, & \text{if } A_i = \emptyset \text{ and } E_C(A) \neq \emptyset, \\ \{W_i(A) \cup U_i^c(A,P) : P \subset E_C(A)\}, & \text{if } A_i \neq \emptyset. \end{cases} \tag{3.4.10}$$

Observe that F is well defined. Indeed, if $U \in F(A)$ is such that $U_i \neq \emptyset$ for i in some $V(C)$, then $U_j \neq \emptyset$ for all $j \in V(C)$. Also, there exists $C \in C(H)$ such that $U_i \neq \emptyset$ for all $i \in V(C)$. Moreover, the values of F are finite, and hence closed.

We show that F is a multivalued G-contraction.

Proposition 3.4.1. *Let $F : X \to X$ be the multivalued map defined above. Then F is a G-contraction.*

Proof. We want to show that F is a G-contraction with constant of contraction

$$\lambda = \max\Bigg\{ \max\{\lambda_{i,j} : (i,j) \in E(H)\}, \max\left\{\frac{R}{R_i} : i \in \{1,\dots,p\}\right\},$$
$$\max\left\{\frac{R_i}{R_j} : i \in V_{\mu_i}, j \in V_{\mu_j} \text{ for } \mu_i, \mu_j \in I \text{ such that } \mu_i < \mu_j\right\}\Bigg\}, \quad (3.4.11)$$

where R_i, I and V_μ for $\mu \in I$ are given in Lemma 3.3.1. For $i, k \in V(C)$ for some $C \in C(H)$, we denote

$$\lambda_{i\to k} = \max\left\{\lambda_{i_0,i_1}\cdots\lambda_{i_{N-1},i_N} : [i_k]_0^N \in \{i \xrightarrow{C} k\}\right\}, \quad (3.4.12)$$

where $\{i \xrightarrow{C} k\}$ is given in (3.4.4). Observe that $\lambda_{i\to k} \le \lambda$.

Let $A, B \in X$ be such that $(A,B) \in E(G)$ and $U \in F(A)$. We look for $\widehat{U} \in F(B)$ such that $(U, \widehat{U}) \in E(G)$ and $d(U, \widehat{U}) \le \lambda d(A,B)$.

Step 1: $\mathbf{i} \in \mathbf{V^e}$: Let $\mu \in I$ be such that $i \in V_\mu$.

Case 1: $U_i = \emptyset$ and $\widetilde{U}_i \neq \emptyset$ for every $\widetilde{U} \in F(B)$:

In this case, $E_i(A) = \emptyset$ and $E_i(B) \neq \emptyset$ by (3.4.9). Choose some $(i,j) \in E_i(B)$. Therefore, $A_j = \emptyset$, $B_j \neq \emptyset$, and for $\nu \in I$ such that $j \in V_\nu$, one has $\mu < \nu$.

By condition (G)(ii)(a), if $j \in V^e$, there exists $l \in V(H)$ such that $(j,l) \in E(H)$ and $A_l \neq \emptyset$. So, $(j,l) \in E_j(A)$.

On the other hand, if $j \in V(C)$ for some $C \in C(H)$, by condition (G)(ii)(b), there exist $k \in V(C)$ and $l \in V(H)$ such that $(k,l) \in E(H)$ and $A_l \neq \emptyset$. So, $(k,l) \in E_C(A)$ and $j, k \in V(C)$.

So, for the case $j \in V^e$ and the case $j \in V^c$, we obtain by (3.4.9) and (3.4.10),

$$U_i = \emptyset,\ \widetilde{U}_i \neq \emptyset \quad \text{and} \quad U_j \neq \emptyset \text{ for some } (i,j) \in V(H). \quad (3.4.13)$$

Moreover, by (3.3.1), (3.3.2) and (3.4.11),

$$\overline{D}_i(U_i, \widetilde{U}_i) = R_i = \frac{R_i}{R_j}\overline{D}_j(A_j, B_j) \le \lambda d(A,B) \quad \forall \widetilde{U} \in F(B). \quad (3.4.14)$$

Case 2: $U_i \neq \emptyset$ and $\widetilde{U}_i = \emptyset$ for every $\widetilde{U} \in F(B)$:

In this case, $E_i(A) \neq \emptyset$ and $E_i(B) = \emptyset$ by (3.4.9). Choose some $(i,j) \in E_i(A)$. Therefore, $A_j \neq \emptyset$, $B_j = \emptyset$, and for $\nu \in I$ such that $j \in V_\nu$, one has $\mu < \nu$. By conditions (G)(i) and (G)(iii), one has $j \in V^e$ and $B_i \neq \emptyset$. By (3.4.9), (3.4.10) and since $B_i \neq \emptyset$, one has

$$\begin{cases} U_i \neq \emptyset,\ \widetilde{U}_i = \emptyset \quad \text{and one of the following situations hold:} \\ \quad \text{- there is no } k \in V(H) \text{ such that } (k,i) \in E(H); \\ \quad \text{- for all } k \in V(H) \text{ such that } (k,i) \in E(H),\ \widetilde{U}_k \neq \emptyset \ \ \forall \widetilde{U} \in F(B). \end{cases} \quad (3.4.15)$$

Also, by (3.3.1), (3.3.2) and (3.4.11),

$$\overline{D}_i(U_i, \widetilde{U}_i) = R_i = \frac{R_i}{R_j} \overline{D}_j(A_j, B_j) \leq \lambda d(A, B) \quad \forall \widetilde{U} \in F(B). \tag{3.4.16}$$

Case 3: $U_i \neq \emptyset$ and $\widetilde{U}_i \neq \emptyset$ for every $\widetilde{U} \in F(B)$:

In this case, $U_i = U_i^e(A, P)$ for some $\emptyset \neq P \subset E_i(A)$, and $E_i(B) \neq \emptyset$ by (3.4.9). If $P \subset E_i(B)$, one has by (3.3.1), (3.4.2) and (3.4.11),

$$\begin{aligned} D_i(U_i^e(A,P), U_i^e(B,P)) &= D_i\left(\bigcup_{(i,j)\in P} T_{i,j}(A_j), \bigcup_{(i,j)\in P} T_{i,j}(B_j)\right) \\ &\leq \max_{(i,j)\in P} D_i\Big(T_{i,j}(A_j), T_{i,j}(B_j)\Big) \\ &\leq \max_{(i,j)\in P} \lambda_{i,j} D_j(A_j, B_j) \\ &\leq \lambda d(A, B). \end{aligned} \tag{3.4.17}$$

If $P \not\subset E_i(B)$, choose some $(i, j) \in P \backslash E_i(B)$. So, $A_j \neq \emptyset$, $B_j = \emptyset$ and, for $\nu \in I$ such that $j \in V_\nu$, one has $\mu < \nu$. Thus, by (3.3.1), (3.3.2) and (3.4.11),

$$\overline{D}_i(U_i, \widetilde{U}_i) \leq R = \frac{R}{R_j} \overline{D}_j(A_j, B_j) \leq \lambda d(A, B) \quad \forall \widetilde{U} \in F(B). \tag{3.4.18}$$

Combining (3.4.17) and (3.4.18), for $U_i = U_i^e(A, P)$ for some $P \subset E_i(A)$, we choose $\widetilde{U}_i \in F_i(B)$ such that

$$\widetilde{U}_i = \begin{cases} U_i^e(B, P), & \text{if } P \subset E_C(A) \cap E_C(B), \\ \widetilde{U}_i, & \text{otherwise, with some } \widetilde{U}_i \in F_i(B); \end{cases} \tag{3.4.19}$$

and we get

$$\overline{D}_i(U_i, \widetilde{U}_i) \leq \lambda d(A, B). \tag{3.4.20}$$

Step 2: $\mathbf{i \in V(C)}$ for some $\mathbf{C \in C(H)}$: Let $\mu \in I$ be such that $i \in V_\mu$.

Case 4: $U_i = \emptyset$ and $\widetilde{U}_i \neq \emptyset$ for every $\widetilde{U} \in F(B)$:

In this case, $A_i = E_C(A) = \emptyset$ and $B_i \cup E_C(B) \neq \emptyset$ by (3.4.10).

If $B_i \neq \emptyset$, by condition (G)(ii)(b), there exist $k \in V(C)$ and $j \in V(H)$ such that $(k, j) \in E(H)$ and $A_j \neq \emptyset$. So, $(k, j) \in E_C(A)$. This contradicts the fact that $E_C(A) = \emptyset$.

If $E_C(B) \neq \emptyset$, by (3.4.3), there exist $k \in V(C)$ and $j \in V(H) \backslash V(C)$ such that $(k, j) \in E(H)$ and $B_j \neq \emptyset$ and, for $\nu \in I$ such that $j \in V_\nu$, one has $\mu < \nu$. Since $E_C(A) = \emptyset$, $A_j = \emptyset$. If $j \in V^e$, by condition (G)(ii)(a), there exists $l \in V(H)$ such that $(j, l) \in E(H)$ and $A_l \neq \emptyset$. So, $E_j(A) \neq \emptyset$ and $U_j \neq \emptyset$ by (3.4.9). On the other hand, if $j \in V(\widehat{C})$ for some $\widehat{C} \in C(H)$, by condition (G)(ii)(b), there exist $m \in V(\widehat{C})$, $l \in V(H)$ such that $(m, l) \in E(H)$ and $A_l \neq \emptyset$. So, $E_{\widehat{C}}(A) \neq \emptyset$

and $U_j \neq \emptyset$ by (3.4.10). Thus, for the case $j \in V^e$ and the case $j \in V^c$, we obtain

$$U_i = \emptyset,\ \widetilde{U}_i \neq \emptyset \quad \text{and} \quad U_j \neq \emptyset \text{ for some } (k,j) \in E_C(B). \tag{3.4.21}$$

Moreover, by (3.3.1), (3.3.2) and (3.4.11),

$$\overline{D}_i(U_i, \widetilde{U}_i) = R_i = \frac{R_i}{R_j}\overline{D}_j(A_j, B_j) \leq \lambda d(A,B) \quad \forall \widetilde{U} \in F(B). \tag{3.4.22}$$

Case 5*: $U_i \neq \emptyset$ and $\widetilde{U}_i = \emptyset$ for every $\widetilde{U} \in F(B)$:*

In this case, $A_i \cup E_C(A) \neq \emptyset$ and $B_i \cup E_C(B) = \emptyset$ by (3.4.10). From condition (G)(iii), we deduce that $A_i = B_i = \emptyset$. Let $(k,j) \in E_C(A)$. One has $A_j \neq \emptyset$ and $B_j = \emptyset$ since $(k,j) \notin E_C(B)$. By condition (G)(iii), $j \in V^e$ and $B_k \neq \emptyset$ since $(k,j) \in E(H)$. This implies that $B_i \neq \emptyset$ by condition (Xii) since $i,k \in V(C)$. This is a contradiction. Thus,

$$U_i \neq \emptyset \text{ and } \widetilde{U}_i = \emptyset \quad \forall \widetilde{U} \in F(B) \ \text{ is impossible.} \tag{3.4.23}$$

Case 6*: $U_i \neq \emptyset$ and $\widetilde{U}_i \neq \emptyset$ for every $\widetilde{U} \in F(B)$:*

In this case, $A_i \cup E_C(A) \neq \emptyset$ and $B_i \cup E_C(B) \neq \emptyset$ by (3.4.10).

If $A_i \neq \emptyset$, by condition (G)(iii), $B_i \neq \emptyset$. So $W_i(A) \neq \emptyset$, $W_i(B) \neq \emptyset$, and, by (3.3.1), (3.4.7), and (3.4.11),

$$\begin{aligned} D_i(W_i(A), W_i(B)) &= D_i\left(\bigcup_{(i,j)\in E(C)} T_{i,j}(A_j), \bigcup_{(i,j)\in E(C)} T_{i,j}(B_j)\right) \\ &\leq \max_{(i,j)\in E(C)} D_i\Big(T_{i,j}(A_j), T_{i,j}(B_j)\Big) \\ &\leq \max_{(i,j)\in E(C)} \lambda_{i,j} D_j(A_j, B_j) \\ &\leq \lambda \max_{(i,j)\in E(C)} D_j(A_j, B_j) \\ &\leq \lambda d(A,B). \end{aligned} \tag{3.4.24}$$

If $E_C(A) \neq \emptyset$, for $\emptyset \neq P \subset E_C(A)$ such that $P \subset E_C(B)$, one has by (3.3.1), (3.4.5), (3.4.6), (3.4.11) and (3.4.12),

$$\begin{aligned} D_i(U_i^c(A,P), U_i^c(B,P)) &= D_i\left(\bigcup_{(k,j)\in P} T_{i\to k}\circ T_{k,j}(A_j), \bigcup_{(k,j)\in P} T_{i\to k}\circ T_{k,j}(B_j)\right) \\ &\leq \max_{(k,j)\in P} \lambda_{i\to k} D_k\Big(T_{k,j}(A_j), T_{k,j}(B_j)\Big) \\ &\leq \max_{(k,j)\in P} \lambda_{i\to k}\lambda_{k,j} D_j(A_j, B_j) \\ &\leq \lambda \max_{(k,j)\in P} D_j(A_j, B_j) \\ &\leq \lambda d(A,B). \end{aligned} \tag{3.4.25}$$

If $P \subset E_C(A)$ and $P \not\subset E_C(B)$, there exists $(k,j) \in P$ such that $A_j \neq \emptyset$, $B_j = \emptyset$ and, for $\nu \in I$ such that $j \in V_\nu$, one has $\mu < \nu$. Hence, by (3.3.1), (3.3.2) and (3.4.11),

$$\overline{D}_i(U_i, \widetilde{U}_i) \leq R = \frac{R}{R_j}\overline{D}_j(A_j, B_j) \leq \lambda d(A,B) \quad \forall \widetilde{U} \in F(B). \tag{3.4.26}$$

Combining (3.4.10), (3.4.24), (3.4.25) and (3.4.26), we choose $\widetilde{U}_i \in F_i(B)$ such that

$$\widetilde{U}_i = \begin{cases} W_i(B), & \text{if } U_i = W_i(A), \\ U_i^c(B,P), & \text{if } U_i = U_i^c(A,P) \\ & \quad \text{for } \emptyset \neq P \subset E_C(A)\cap E_C(B), \\ W_i(B)\cup U_i^c(B,P), & \text{if } U_i = W_i(A)\cup U_i^c(A,P) \\ & \quad \text{for } \emptyset \neq P \subset E_C(A)\cap E_C(B), \\ \widetilde{U}_i, & \text{otherwise, with some } \widetilde{U}_i \in F_i(B); \end{cases} \tag{3.4.27}$$

and we get

$$\overline{D}_i(U_i, \widetilde{U}_i) \leq \lambda d(A,B). \tag{3.4.28}$$

Step 3: Choice of an appropriate $\widetilde{\mathbf{U}} \in \mathbf{F}(\mathbf{B})$:

Finally, we choose $\widetilde{U} \in F(B)$ as follows:

$$\widetilde{U}_i = \begin{cases} \emptyset, & \text{if } i \in V^e,\ E_i(B) = \emptyset, \\ \text{some } \widetilde{U}_i \in F_i(B), & \text{if } i \in V^e,\ U_i = \emptyset,\ E_i(B) \neq \emptyset, \\ \widetilde{U}_i \text{ given by (3.4.19)}, & \text{if } i \in V^e,\ U_i \neq \emptyset,\ E_i(B) \neq \emptyset, \\ \emptyset, & \text{if } i \in V(C),\ B_i \cup E_C(B) = \emptyset, \\ \text{some } \widetilde{U}_i \in F_i(B), & \text{if } i \in V(C),\ U_i = \emptyset, \\ \widetilde{U}_i \text{ given by (3.4.27)}, & \text{if } i \in V(C),\ U_i \neq \emptyset,\ B_i \cup E_C(B) \neq \emptyset. \end{cases} \tag{3.4.29}$$

It follows from (3.4.13), (3.4.15), (3.4.21) and (3.4.23) that

$$(U, \widetilde{U}) \in E(G).$$

Finally, from (3.4.14), (3.4.16), (3.4.20) (3.4.22) and (3.4.28), we deduce that

$$d(U, \widetilde{U}) \leq \lambda d(A, B).$$

Therefore, F is a G-contraction. □

Here is an other property satisfied by the multivalued map F.

Lemma 3.4.1. *Let $F : X \to X$ be the multivalued map defined above. Then, for every $A^0 \in X$ and every $\{A^n\}$ G_1-Picard trajectory from A^0 converging to some $A \in X$, there exists $N \in \mathbb{N}$ such that $(A^n, A) \in E(G)$ for all $n > N$.*

Proof. Let $A^0 \in X$ and $\{A^n\}$ a G_1-Picard trajectory from A^0 such that $A^n \to A$. Thus, there exists $N \in \mathbb{N}$ such that $d(A^n, A) < R$ for all $n > N$. So, by (3.3.1) and (3.3.2), $A^n = (A_1^n, \ldots, A_p^n)$ and $A = (A_1, \ldots, A_p)$ are such that, for all $n > N$ and all $i \in V(H)$, $A_i^n = \emptyset$ if and only if $A_i = \emptyset$. Thus, (G)(i) is satisfied and $(A^n, A) \in E(G)$ for all $n > N$. □

3.5. Attractor of an H-IFS and elements of $C(H)$

For $H = (V(H), E(H))$ a MW-directed graph, and $\{T_{i,j}\}_H$ a graph-directed iterated function system over the graph H, we consider K the attractor of this H-IFS insured by Theorem 3.2.1. We want to get more information on K by taking into account the connected components of H.

Theorem 3.5.1. *Let $H = (V(H), E(H))$ be a MW-directed graph. Let $\{T_{i,j}\}_H$ be an H-IFS and K its attractor. Then the following statements hold:*

(1) For every $C \in C(H)$, there exists $K^+(C) \subset K$ such that

(a) $K_i^+(C) \neq \emptyset$ for every $i \in V(C)$;

(b) $K_i^+(C) \neq \emptyset$ *for every* $i \in [C]_{\leftarrow}$, *where* $[C]_{\leftarrow}$ *is defined in* (3.2.2).

(c) $K_i^+(C) = \emptyset$ *for every* $i \notin [C]_{\leftarrow}$.

(2) If $C_1, C_2 \in C(H)$ *are such that* $C_1 \preceq C_2$*, then* $K^+(C_1) \subset K^+(C_2)$.

(3) If $C_1, C_2 \in C(H)$ *are incomparable, then,*

$$K_i^+(C_1) \cap K_i^+(C_2) = \emptyset \quad \forall i \notin \Big([C_1]_{\leftarrow}\Big) \cap \Big([C_2]_{\leftarrow}\Big).$$

(4) There exists $K^- \in X$ *such that* $K^- \subset K$ *and*

(a) for every $C \in C(H)$, $K_i^- = K_i^+(C)$ *for every* $i \in V(C)$ *and* $K_i^- \subset K_i^+(C)$ *for every* $i \in [C]_{\leftarrow}$*;*

(b) If $C_1, C_2 \in C(H)$ *are such that* $C_1 \preceq C_2$*, then*

$$K_i^- \subset K_i^+(C_1) \subset K_i^+(C_2) \quad \forall i \in [C_1]_{\leftarrow};$$

(c) If $C_1, C_2 \in C(H)$ *are incomparable, then,*

$$K_i^- \subset K_i^+(C_1) \cap K_i^+(C_2) \quad \forall i \in \Big([C_1]_{\leftarrow}\Big) \cap \Big([C_2]_{\leftarrow}\Big).$$

Proof. (1) Let $F : X \to X$ be the multivalued map defined in (3.4.8), (3.4.9) and (3.4.10). We know that F is a G-contraction by Proposition 3.4.1. Also, it follows from Lemma 3.4.1 that F satisfies condition (ii) of Theorem 3.2.2.

Theorem 3.2.1 and the definition of F imply that fixed points of F are included in K.

Let $C \in C(H)$. We want to show that there exists $K^+(C)$ a fixed point of F satisfying the required properties. Fix

$$A^0 = (A_1^0, \dots, A_p^0) \in X \quad \text{such that} \quad A_i^0 \neq \emptyset \iff i \in V(C). \tag{3.5.1}$$

For $n \in \mathbb{N} \cup \{0\}$, we choose inductively

$$A^{n+1} \in F(A^n) \quad \text{the biggest element of } F(A^n), \tag{3.5.2}$$

that is, by (3.4.9) and (3.4.10), $A^{n+1} = (A_1^{n+1}, \dots, A_p^{n+1}) \in F(A^n)$ is chosen as follows:

For $i \in V^e$,

$$A_i^{n+1} = \begin{cases} \emptyset, & \text{if } E_i(A^n) = \emptyset; \\ U_i^e\Big(A^n, E_i(A^n)\Big), & \text{if } E_i(A^n) \neq \emptyset, \end{cases} \tag{3.5.3}$$

where E_i^e and U_i^e are defined in (3.4.1) and (3.4.2) respectively.

For $i \in V(\widehat{C})$ for some $\widehat{C} \in C(H)$,

$$A_i^{n+1} = \begin{cases} \emptyset, & \text{if } A_i^n = E_{\widehat{C}}(A^n) = \emptyset; \\ U_i^c\Big(A^n, E_{\widehat{C}}(A^n)\Big), & \text{if } A_i^n = \emptyset,\ E_{\widehat{C}}(A^n) \neq \emptyset; \\ W_i(A^n) \cup U_i^c\Big(A^n, E_{\widehat{C}}(A^n)\Big), & \text{if } A_i^n \neq \emptyset; \end{cases} \tag{3.5.4}$$

where $E_{\widehat{C}}$, U_i^c and W_i are defined in (3.4.3), (3.4.6) and (3.4.7) respectively.

Arguing as in the proof of Proposition 3.4.1, one has that $(A^{n-1}, A^n) \in E(G)$ and

$$d(A^n, A^{n+1}) \leq \lambda d(A^{n-1}, A^n) \quad \forall n \in \mathbb{N}.$$

By Theorem 3.2.2, $\{A^n\}$ is a G_1-Picard trajectory converging to some $K^+(C) \in X$ a fixed point of F.

Observe that, for every $n \in \mathbb{N}$ and every $i \in V(C)$, $A_i^n \neq \emptyset$. Therefore,

$$K_i^+(C) \neq \emptyset \quad \forall i \in V(C).$$

Similarly, observe that, by construction, $A_i^n = \emptyset$ for every $i \notin [C]_{\leftarrow}$. Indeed, for such i, $E_i(A^{n-1}) = \emptyset$ if $i \in V^e$, and $A_i^{n-1} = E_{\widehat{C}}(A^{n-1}) = \emptyset$ if $i \in V(\widehat{C})$ for some $\widehat{C} \in V(C)$. Thus,

$$K_i^+(C) = \emptyset \quad \forall i \notin [C]_{\leftarrow}.$$

On the other hand, let

$$N_C = \max_{i \in [C]_{\leftarrow}} \Big\{ \min\{N : i = i_0, i_N \in V(C), \; [i_k]_0^N \text{ is a path in } H \text{ from } i \text{ to } i_N\} \Big\}. \tag{3.5.5}$$

Again by construction, $A_i^n \neq \emptyset$ for all $n > N_C$, for all $i \in [C]_{\leftarrow}$. So,

$$K_i^+(C) \neq \emptyset \quad \forall i \in [C]_{\leftarrow}.$$

Finally, observe that $K^+(C)$ is independent of $A^0 \subset X$ chosen as in (3.5.1). Indeed, for

$$\widetilde{A}^0 = (\widetilde{A}_1^0, \dots, \widetilde{A}_p^0) \in X \quad \text{such that} \quad \widetilde{A}_i^0 \neq \emptyset \iff i \in V(C),$$

we define inductively $\widetilde{A}^{n+1} \in F(\widetilde{A}^n)$ as in (3.5.2). Observe that $(A^n, \widetilde{A}^n) \in E(G)$ for all $n \in \mathbb{N} \cup \{0\}$. Arguing as in Proposition 3.4.1, one has

$$d(A^{n+1}, \widetilde{A}^{n+1}) \leq \lambda d(A^n, \widetilde{A}^n) \quad \forall n \in \mathbb{N}.$$

This inequality combined with the fact that $A^n \to K^+(C)$ imply that $\widetilde{A}^n \to K^+(C)$.

(2) Let $C_1, C_2 \in C(H)$ be such that $C_1 \preceq C_2$. One has

$$\Big\{i \in [C_1]_{\leftarrow}\Big\} \subset \Big\{i \in [C_2]_{\leftarrow}\Big\}.$$

Let $B^0 = (B_1^0, \dots, B_p^0) \in X$ be such that

$$B_j^0 = \begin{cases} K_j^+(C_2), & \text{if } j \in [C_1]_{\leftarrow}, \\ \emptyset, & \text{if } j \notin [C_1]_{\leftarrow}. \end{cases}$$

By (1) and (G)(i), one has $(K^+(C_1), B^0) \in E(G)$ and $K^+(C_1) \in F(K^+(C_1))$. Let B^1 be the biggest element in $F(B^0)$, i.e. B^1 is chosen similarly to (3.5.3) and (3.5.4). Observe that $B^1 \subset K^+(C_2)$, since $B^0 \subset K^+(C_2)$, $K^+(C_2) \in F(K^+(C_2))$, and by the definitions of F and $K^+(C_2)$. Arguing as in the proof of Proposition 3.4.1, one has $(K^+(C_1), B^1) \in E(G)$ and

$$d(K^+(C_1), B^1) \leq \lambda d(K^+(C_1), B^0).$$

Repeating this argument, we obtain $\{B^n\}$ a G_1-Picard trajectory from B^0 such that

$$B^n \subset K^+(C_2) \quad \text{and} \quad d(K^+(C_1), B^n) \leq \lambda^n d(K^+(C_1), B^0) \quad \forall n \in \mathbb{N}.$$

Therefore, $B^n \to K^+(C_1)$ and

$$K^+(C_1) \subset K^+(C_2).$$

(3) If $C_1, C_2 \in C(H)$ are incomparable, it follows directly from (1)(c) that

$$K_i^+(C_1) \cap K_i^+(C_2) = \emptyset \quad \forall i \notin \Big([C_1]_{\leftarrow}\Big) \cap \Big([C_2]_{\leftarrow}\Big).$$

(4) For every $C \in C(H)$, $C = (V(C), E(C))$ is a MW-directed graph and

$$\{T_{i,j} : (i,j) \in E(C)\}$$

is a graph-directed iterated function system over the graph C. Let

$$K^-(C) = (K_i^-)_{i \in V(C)}$$

be the attractor of this graph-directed iterated system insured by Theorem 3.2.1.

We define $K^- \in X$ by

$$K^- = (K_1^-, \dots, K_p^-),$$

$$\text{where} \quad K_i^- = \begin{cases} K_i^-(C), & \text{if } i \in V(C) \text{ for some } C \in C(H), \\ \emptyset, & \text{if } i \in V^e. \end{cases} \tag{3.5.6}$$

Let $C \in C(H)$ and $\{A^n\}$ the G_1-Picard trajectory from A^0 defined in (3.5.1) and (3.5.2). By (3.5.4), for all $n \in \mathbb{N}$, $E_C(A^{n-1}) = \emptyset$ and $A_i^n = W_i(A^{n-1})$ for all $i \in V(C)$. So, using (3.4.7), (3.4.10) and the fact that $A_i^n \to K_i^+(C) \in F_i(K^+(C))$ for every $i \in V(C)$, we deduce that

$$K_i^+(C) = \bigcup_{(i,j)\in E(C)} T_{i,j}(K_j^+(C)) \quad \forall i \in V(C).$$

By definition of K^-,

$$K_i^- = \bigcup_{(i,j)\in E(C)} T_{i,j}(K_j^-) \quad \forall i \in V(C).$$

The uniqueness of the fixed point of this operator implies that

$$K_i^+(C) = K_i^- \quad \forall i \in V(C). \tag{3.5.7}$$

On the other hand, if $i \in V^e \cap [C]_{\leftarrow}$, one has $\emptyset = K_i^- \subset K_i^+(C)$. If $i \in V(\widehat{C}) \cap [C]_{\leftarrow}$ for some $C \neq \widehat{C} \in C(H)$, then $\widehat{C} \preceq C$. It follows from (3.5.7) and (2), that $K_i^- = K_i^+(\widehat{C}) \subset K_i^+(C)$.

The properties (4)(b) and (4)(c) follow directly from (2) and (4)(a). $\square$

Example 3.5.1. *Let H be the MW-graph of Figure 3.3.*

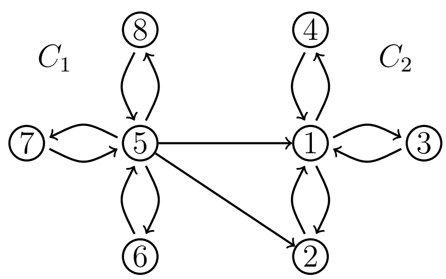

FIG. 3.3. An MW-graph H with $C(H) = \{C_1, C_2\}$.

We consider the H-IFS, $\{T_{i,j}\}_H$, with the metric spaces:

$$X_1 = [1,2]\times[0,1], \qquad X_2 = [2,3]\times[0,1], \qquad X_3 = [1,2]\times[1,2],$$
$$X_4 = [2,3]\times[1,2], \qquad X_5 = [0,1]\times[0,1], \qquad X_6 = [-1,0]\times[0,1],$$
$$X_7 = [0,1]\times[1,2], \qquad X_8 = [-1,0]\times[1,2],$$

and the contractions:

$$T_{1,2}(x) = M_1 x + \left(\frac{-2}{5}, \frac{1}{5}\right), \qquad T_{1,3}(x) = M_1 x + \left(\frac{1}{5}, \frac{-4}{5}\right),$$
$$T_{1,4}(x) = M_3 x + \left(\frac{-1}{3}, \frac{-1}{3}\right), \qquad T_{2,1}(x) = M_2 x + \left(\frac{14}{8}, \frac{3}{8}\right),$$
$$T_{3,1}(x) = M_2 x + \left(\frac{3}{8}, 1\right), \qquad T_{4,1}(x) = M_4 x + \left(\frac{5}{4}, \frac{5}{4}\right),$$

$$T_{5,1}(x) = M_4 x + \left(\frac{-2}{4}, \frac{1}{4}\right), \qquad T_{5,2}(x) = M_3 x + \Big(-1, 0\Big),$$
$$T_{5,6}(x) = M_1 x + \Big(1, 0\Big), \qquad T_{5,7}(x) = M_1 x + \left(0, \frac{-3}{5}\right),$$
$$T_{5,8}(x) = M_3 x + \left(\frac{2}{3}, \frac{-2}{3}\right), \qquad T_{6,5}(x) = M_2 x + \left(\frac{-5}{8}, 0\right),$$
$$T_{7,5}(x) = M_2 x + \left(0, \frac{11}{8}\right), \qquad T_{8,5}(x) = M_4 x + \Big(-1, 1\Big),$$

where

$$M_1 = \begin{pmatrix} \frac{4}{5} & 0 \\ 0 & \frac{4}{5} \end{pmatrix}, \quad M_2 = \begin{pmatrix} \frac{5}{8} & 0 \\ 0 & \frac{5}{8} \end{pmatrix}, \quad M_3 = \begin{pmatrix} \frac{2}{3} & 0 \\ 0 & \frac{2}{3} \end{pmatrix}, \quad M_4 = \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{3}{4} \end{pmatrix}.$$

Figures 3.4 and 3.5 present $K^+(C_2)$ and K^- respectively.

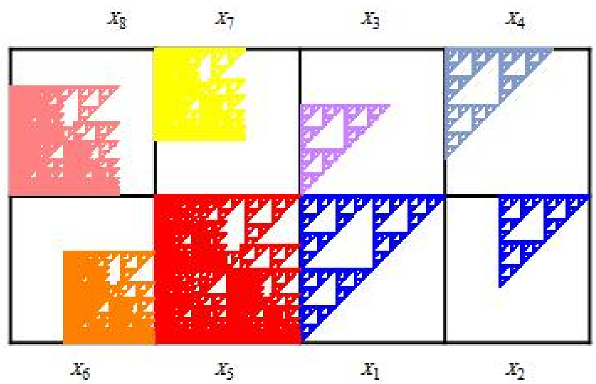

FIG. 3.4. The set $K^+(C_2)$.

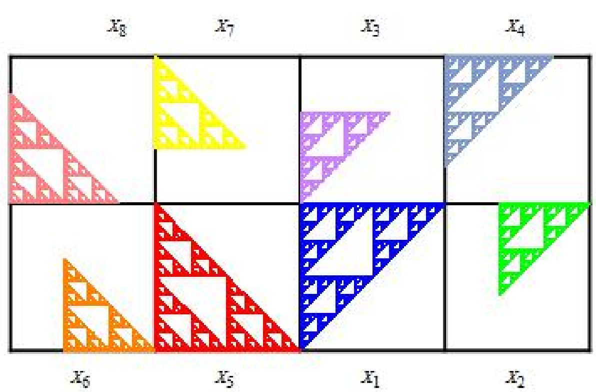

FIG. 3.5. The set K^-.

3.6. Attractor of an H-IFS and subsets of $C(H)$

We obtain other information on the attractor of the graph-directed iterated function system by considering subsets of $C(H)$.

Theorem 3.6.1. *Let $H = (V(H), E(H))$ be a MW-directed graph. Let $\{T_{i,j}\}_H$ be an H-IFS and K its attractor. Then the following statements hold:*

(1) For every $\mathcal{S} \subset C(H)$, there exists $K^+(\mathcal{S}) \subset K$ such that
   (a) $K^+(C) \subset K^+(\mathcal{S})$ for every $C \in \mathcal{S}$;
   (b) $K_i^+(C) = K_i^+(\mathcal{S})$ for every $i \in V(C)$ and every maximal element $C \in \mathcal{S}$;
   (c) $K_i^+(\mathcal{S}) \neq \emptyset$ if and only if $i \in \bigcup_{C\in\mathcal{S}}[C]_{\leftarrow}$.

(2) If $\mathcal{S}_1, \mathcal{S}_2 \subset C(H)$ are such that for every $C_1 \in \mathcal{S}_1$, there exists $C_2 \in \mathcal{S}_2$ such that $C_1 \preceq C_2$, then $K^+(\mathcal{S}_1) \subset K^+(\mathcal{S}_2)$.

(3) For $\mathcal{S}_1, \mathcal{S}_2 \subset C(H)$, one has

$$K^+(\mathcal{S}_1) \cap K^+(\mathcal{S}_2) = \emptyset \quad \textit{if} \quad \Big(\bigcup_{C\in\mathcal{S}_1}[C]_{\leftarrow}\Big) \cap \Big(\bigcup_{C\in\mathcal{S}_2}[C]_{\leftarrow}\Big) = \emptyset.$$

(4) The attractor K is such that $K = K^+(C(H))$.

Proof. (1) By Proposition 3.4.1 and Lemma 3.4.1, the map $F : X \to X$ defined in (3.4.8), (3.4.9) and (3.4.10) is a G-contraction satisfying condition (ii) of Theorem 3.2.2. Also, from the proof of Theorem 3.5.1, we know that fixed points of F are included in K.

Let $\mathcal{S} \subset C(H)$. We want to show that there exists $K^+(\mathcal{S})$ a fixed point of F satisfying the required properties. Fix

$$\begin{cases} \widehat{A}^0 = (\widehat{A}_1^0, \dots, \widehat{A}_p^0) \in X \quad \text{such that} \quad \widehat{A}_i^0 \neq \emptyset \iff i \in \bigcup_{C\in\mathcal{S}} V(C), \\ \text{and} \quad \widehat{A}_i^0 = A_i^0 \quad \text{if } i \in V(C) \text{ for } C \in \mathcal{S}, \text{ where } A^0 \text{ is defined in (3.5.1)}. \end{cases} \tag{3.6.1}$$

For $n \in \mathbb{N} \cup \{0\}$, we choose inductively

$$\widehat{A}^{n+1} \in F(\widehat{A}^n) \quad \text{the biggest element of } F(\widehat{A}^n), \tag{3.6.2}$$

Arguing as in the proof of Theorem 3.5.1, one deduces that $\{\widehat{A}^n\}$ is a G_1-Picard trajectory converging to some $K^+(\mathcal{S}) \in X$ a fixed point of F. Also, $K^+(\mathcal{S})$ is independent of $\widehat{A}^0$ chosen as in (3.6.1).

For $C \in \mathcal{S}$, observe that $A^n \subset \widehat{A}^n$ for all $n \in \mathbb{N} \cup \{0\}$, where A^n is defined in (3.5.1) and (3.5.2). Since $\widehat{A}^n \to K^+(\mathcal{S})$ and $A^n \to K^+(C)$, we deduce that

$$K^+(C) \subset K^+(\mathcal{S}).$$

It follows from this inclusion and Theorem 3.5.1(1)(b) that

$$K_i^+(\mathcal{S}) \neq \emptyset \quad \forall i \in \bigcup_{C\in\mathcal{S}}[C]_{\leftarrow}.$$

On the other hand,

$$\widehat{A}_i^n = \emptyset \quad \forall i \notin \bigcup_{C \in \mathcal{S}} [C]_{\leftarrow}, \quad \forall n \in \mathbb{N}.$$

Thus, (1)(c) holds.

In the particular case where $C \in \mathcal{S}$ is maximal, one has

$$A_i^n = \widehat{A}_i^n \quad \forall i \in V(C), \forall n \in \mathbb{N} \cup \{0\},$$

where A^n is defined in (3.5.2). Since

$$A_i^n \to K_i^+(C) \quad \text{and} \quad \widehat{A}_i^n \to K_i^+(\mathcal{S}),$$

one has

$$K_i^+(C) = K_i^+(\mathcal{S}) \quad \forall i \in V(C).$$

(2) and (3): The proofs are respectively analogous to those of (2) and (3) in Theorem 3.5.1.

(4) Let $\mathcal{S} = C(H)$. Since $K^+(C(H))$ is independent of the choice of $\widehat{A}^0$ in (3.6.1), we can fix

$$\widehat{A}^0 = (\widehat{A}_1^0, \dots, \widehat{A}_p^0) \in X \quad \text{such that} \quad \widehat{A}_i^0 = \begin{cases} K_i, & \text{if } i \in V^c, \\ \emptyset, & \text{if } i \in V^e, \end{cases}$$

where V^c and V^e are defined in (3.3.3) and (3.3.4) respectively. Let $\widehat{A}^n$ be defined as in (3.6.2). We know that $\widehat{A}^n \to K^+(C(H))$. On the other hand, since K is the unique attractor of this H-IFS obtained in Theorem 3.2.1, we deduce that $K = K^+(C(H))$. $\square$

In the following result, we see that the maximal elements of $C(H)$ play a key role.

Corollary 3.6.1. *Let $H = (V(H), E(H))$ be a MW-directed graph, and $\{T_{i,j}\}_H$ an H-IFS. Then, for every $\mathcal{S}_1, \mathcal{S}_2 \subset C(H)$ such that*

$$\begin{aligned} &\{C \in \mathcal{S}_1 : C \textit{ is a maximal element of } \mathcal{S}_1\} \\ =&\{C \in \mathcal{S}_2 : C \textit{ is a maximal element of } \mathcal{S}_2\}, \end{aligned}$$

one has

$$K^+(\mathcal{S}_1) = K^+(\mathcal{S}_2).$$

Proof. Let $\mathcal{S} \subset C(H)$ and let

$$\mathcal{S}_m = \{C \in \mathcal{S} : C \text{ is a maximal element of } \mathcal{S}\}.$$

To conclude, it is sufficient to show that

$$K^+(\mathcal{S}) = K^+(\mathcal{S}_m).$$

It follows from Theorem 3.6.1(2) that

$$K^+(\mathcal{S}) \subset K^+(\mathcal{S}_m) \quad \text{and} \quad K^+(\mathcal{S}_m) \subset K^+(\mathcal{S}).$$

□

3.7. Other fixed points of our G-contraction

In the proofs of Theorems 3.5.1 and 3.6.1, $K^+(C)$ and $K^+(\mathcal{S})$ were obtained as fixed point of the multivalued G-contraction F. In fact, much more fixed points of F can be obtained in order to get more information on the attractor K.

Let $\mathcal{S} \subset C(H)$. For a vertex $i \in V^e$, we consider the set of edges from i on a path to some vertex in $\mathcal{S}$

$$\mathcal{E}_i(\mathcal{S}) = \begin{cases} \emptyset, & \text{if } i \notin \bigcup_{C \in \mathcal{S}} [C]_{\leftarrow}, \\ \left\{ (i,j) \in E(H) : i, j \in \bigcup_{C \in \mathcal{S}} [C]_{\leftarrow} \right\}, & \text{otherwise.} \end{cases}$$

Similarly, for $\widehat{C} \in C(H)$, we consider

$$\mathcal{E}_{\widehat{C}}(\mathcal{S}) = \begin{cases} \emptyset, & \text{if } V(\widehat{C}) \not\subset \bigcup_{C \in \mathcal{S}} [C]_{\leftarrow}, \\ \left\{ (i,j) \in E(H) : i \in V(\widehat{C}), j \notin V(\widehat{C}), j \in \bigcup_{C \in \mathcal{S}} [C]_{\leftarrow} \right\}, & \text{otherwise.} \end{cases}$$

Finally, we consider suitable subsets of edges on paths in H reaching $\mathcal{S}$, i.e. subsets of $\mathcal{E}_i(\mathcal{S})$ and $\mathcal{E}_{\widehat{C}}(\mathcal{S})$,

$$\mathcal{Q}(\mathcal{S}) = \Big\{ Q = (Q_i)_{i \in V^e} \times (Q_{\widehat{C}})_{\widehat{C} \in C(H)} : Q_{\widehat{C}} \subset \mathcal{E}_{\widehat{C}}(\mathcal{S}) \ \forall \widehat{C} \in C(H), \quad \text{and}$$
$$\forall i \in V^e, \ Q_i \subset \mathcal{E}_i(\mathcal{S}) \text{ and } Q_i \neq \emptyset \text{ if } \mathcal{E}_i(\mathcal{S}) \neq \emptyset \Big\}. \quad (3.7.1)$$

Using $\mathcal{Q}(\mathcal{S})$, we can obtain more information on $K^+(\mathcal{S})$.

Theorem 3.7.1. *Let $H = (V(H), E(H))$ be a MW-directed graph, and $\{T_{i,j}\}_H$ an H-IFS. Then, the following statements hold:*

(1) For every $\mathcal{S} \subset C(H)$ and every $Q \in \mathcal{Q}(\mathcal{S})$, there exists $K(\mathcal{S}, Q) \in X$ such that
   - *(a) $K(\mathcal{S}, Q) \subset K^+(\mathcal{S})$;*
   - *(b) $K_i(\mathcal{S}, Q) \neq \emptyset$ if and only if $i \in \bigcup_{C \in \mathcal{S}} [C]_{\leftarrow}$;*

(c) $K_i(\mathcal{S}, Q) = K_i^+(\mathcal{S})$ *for every* $i \in V(C)$ *and every* $C \in \mathcal{S}$ *maximal element in* $\mathcal{S}$*.*

(2) For every $\mathcal{S} \subset C(H)$*, if* $Q, \widehat{Q} \in \mathcal{Q}(\mathcal{S})$ *are such that* $Q \subset \widehat{Q}$*, then* $K(\mathcal{S}, Q) \subset K(\mathcal{S}, \widehat{Q})$*.*

(3) Let $\mathcal{S}_1, \mathcal{S}_2 \subset C(H)$ *be such that* $\mathcal{S}_1 \subset \mathcal{S}_2$*. If* $Q \in \mathcal{Q}(\mathcal{S}_1) \cap \mathcal{Q}(\mathcal{S}_2)$*, then* $K(\mathcal{S}_1, Q) \subset K(\mathcal{S}_2, Q)$*.*

(4) Let $\mathcal{S}_1, \mathcal{S}_2 \subset C(H)$ *be such that for every* $C_1 \in \mathcal{S}_1$*, there exists* $C_2 \in \mathcal{S}_2$ *such that* $C_1 \preceq C_2$*. If* $Q^1 \in \mathcal{Q}(\mathcal{S}_1)$ *and* $Q^2 \in \mathcal{Q}(\mathcal{S}_2)$ *are such that* $Q^1 \subset Q^2$*, then* $K(\mathcal{S}_1, Q^1) \subset K(\mathcal{S}_2, Q^2)$*.*

(5) For every $\mathcal{S} \subset C(H)$ *and every* $Q \in \mathcal{Q}(\mathcal{S})$*,* $K_i^- \subset K_i(\mathcal{S}, Q)$ *for every* $i \in V(\widehat{C})$ *and every* $\widehat{C} \in C(H)$ *such that* $V(\widehat{C}) \subset \bigcup_{C \in \mathcal{S}} [C]_{\leftarrow}$*.*

Proof. (1) Let $Q \in \mathcal{Q}(\mathcal{S})$. From Proposition 3.4.1 and Lemma 3.4.1, $F : X \to X$ the multivalued map defined in (3.4.8), (3.4.9) and (3.4.10) is a G-contraction satisfying condition (ii) of Theorem 3.2.2. We want to show that there exists $K(\mathcal{S}, Q)$ a fixed point of F satisfying the required properties.

Fix

$$A^n(\mathcal{S}, Q) = \widehat{A}^n \in X \quad \forall n = 0, \dots, p, \tag{3.7.2}$$

where $\widehat{A}^n$ is defined in (3.6.1) and (3.6.2). From the definition of F, we can observe that

$$A^p(\mathcal{S}, Q) = \Big(A_1^p(\mathcal{S}, Q), \dots, A_p^p(\mathcal{S}, Q)\Big) \in X$$

is such that

$$A_i^p(\mathcal{S}, Q) \neq \emptyset \quad \iff \quad i \in \bigcup_{C \in \mathcal{S}} [C]_{\leftarrow}.$$

Moreover, for every $i \in V^e$,

$$\begin{aligned} Q_i &\subset E_i\Big(A^p(\mathcal{S}, Q)\Big) && \forall i \in \bigcup_{C \in \mathcal{S}} [C]_{\leftarrow}, \\ Q_i &= E_i\Big(A^p(\mathcal{S}, Q)\Big) = \emptyset && \forall i \notin \bigcup_{C \in \mathcal{S}} [C]_{\leftarrow}, \end{aligned}$$

where $E_i\Big(A^p(\mathcal{S}, Q)\Big)$ is defined in (3.4.1). Similarly, for every $\widehat{C} \in C(H)$,

$$\begin{aligned} Q_{\widehat{C}} &\subset E_{\widehat{C}}\Big(A^p(\mathcal{S}, Q)\Big) && \text{if } V(\widehat{C}) \subset \bigcup_{C \in \mathcal{S}} [C]_{\leftarrow}, \\ Q_{\widehat{C}} &= E_{\widehat{C}}\Big(A^p(\mathcal{S}, Q)\Big) = \emptyset && \text{if } V(\widehat{C}) \not\subset \bigcup_{C \in \mathcal{S}} [C]_{\leftarrow}, \end{aligned}$$

where $E_{\widehat{C}}\Big(A^p(\mathcal{S}, Q)\Big)$ is defined in (3.4.3).

For $n > p$, we choose inductively

$$A^n(\mathcal{S},Q) = \Big(A_1^n(\mathcal{S},Q),\dots,A_p^n(\mathcal{S},Q)\Big) \in F\Big(A^{n-1}(\mathcal{S},Q)\Big) \tag{3.7.3}$$

with

$$A_i^n(\mathcal{S},Q) = \begin{cases} \emptyset, & \text{if } i \notin \bigcup_{C\in\mathcal{S}}[C]_\leftarrow, \\ U_i^e\Big(A^{n-1}(\mathcal{S},Q),Q_i\Big), & \text{if } i \in V^e \cap \bigcup_{C\in\mathcal{S}}[C]_\leftarrow, \\ W_i\Big(A^{n-1}(\mathcal{S},Q)\Big) & \\ \ \cup U_i^c\Big(A^{n-1}(\mathcal{S},Q),Q_{\widehat{C}}\Big), & \text{if } \widehat{C} \in C(H) \text{ and} \\ & i \in V(\widehat{C}) \cap \bigcup_{C\in\mathcal{S}}[C]_\leftarrow, \end{cases} \tag{3.7.4}$$

where U_i^e, U_i^c and W_i are defined in (3.4.2), (3.4.6) and (3.4.7) respectively.

Arguing as in the proof of Theorem 3.5.1, one deduces that $\Big\{A^n(\mathcal{S},Q)\Big\}$ is a G_1-Picard trajectory converging to some $K(\mathcal{S},Q) \in X$ a fixed point of F. So, $K(\mathcal{S},Q)$ satisfies (1)(b). Again, it can be shown that $K(\mathcal{S},Q)$ is independent of $A^0(\mathcal{S},Q)$ chosen as in (3.7.2).

Observe that

$$A^n(\mathcal{S},Q) = \Big(A_1^n(\mathcal{S},Q),\dots,A_p^n(\mathcal{S},Q)\Big) \subset \widehat{A}^n = \Big(\widehat{A}_1^n,\dots,\widehat{A}_p^n\Big) \quad \forall n,$$

where $\widehat{A}^n$ is defined in (3.6.2) and $\widehat{A}^n \to K^+(\mathcal{S})$. Moreover, for every C maximal element in $\mathcal{S}$, $\mathcal{E}_C(\mathcal{S}) = \emptyset$ and

$$A_i^n(\mathcal{S},Q) = \widehat{A}_i^n \quad \forall i \in V(C).$$

Therefore $K(\mathcal{S},Q)$ satisfies (1)(a),(c).

(2) Let $Q,\widehat{Q} \in \mathcal{Q}(\mathcal{S})$ be such that $Q \subset \widehat{Q}$. From (3.7.3) and (3.7.4), one sees that

$$A^n(\mathcal{S},Q) \subset A^n(\mathcal{S},\widehat{Q}) \quad \forall n \in \mathbb{N}.$$

Since $A^n(\mathcal{S},Q) \to K(\mathcal{S},Q)$ and $A^n(\mathcal{S},\widehat{Q}) \to K(\mathcal{S},\widehat{Q})$, one has that

$$K(\mathcal{S},Q) \subset K(\mathcal{S},\widehat{Q}).$$

(3) Let $\mathcal{S}_1,\mathcal{S}_2 \subset C(H)$ be such that $\mathcal{S}_1 \subset \mathcal{S}_2$ and let $Q \in \mathcal{Q}(\mathcal{S}_1) \cap \mathcal{Q}(\mathcal{S}_2)$. From (3.7.3) and (3.7.4), one sees that

$$A^n(\mathcal{S}_1,Q) \subset A^n(\mathcal{S}_2,Q) \quad \forall n \in \mathbb{N}.$$

Since $A^n(\mathcal{S}_1,Q) \to K(\mathcal{S}_1,Q)$ and $A^n(\mathcal{S}_2,Q) \to K(\mathcal{S}_2,Q)$, one has that

$$K(\mathcal{S}_1,Q) \subset K(\mathcal{S}_2,Q).$$

(4) Let $\mathcal{S}_1, \mathcal{S}_2 \subset C(H)$ be such that for every $C_1 \in \mathcal{S}_1$, there exists $C_2 \in \mathcal{S}_2$ such that $C_1 \preceq C_2$. One has

$$\Big\{ i \in \bigcup_{C_1 \in \mathcal{S}_1} [C_1]_{\leftarrow} \Big\} \subset \Big\{ i \in \bigcup_{C_2 \in \mathcal{S}_2} [C_2]_{\leftarrow} \Big\}.$$

Let $Q^1 \in \mathcal{Q}(\mathcal{S}_1)$ and $Q^2 \in \mathcal{Q}(\mathcal{S}_2)$ be such that $Q^1 \subset Q^2$. Fix

$$B^p(\mathcal{S}_1, Q^1) = \Big(B_1^p(\mathcal{S}_1, Q^1), \ldots, B_p^p(\mathcal{S}_1, Q^1) \Big) \in X$$

be such that

$$B_j^p(\mathcal{S}_1, Q^1) = \begin{cases} K_j(\mathcal{S}_2, Q^2), & \text{if } j \in \bigcup_{C_1 \in \mathcal{S}_1} [C_1]_{\leftarrow}, \\ \emptyset, & \text{if } j \notin \bigcup_{C_1 \in \mathcal{S}_1} [C_1]_{\leftarrow}. \end{cases}$$

One has $\Big(K(\mathcal{S}_1, Q^1), B^p(\mathcal{S}_1, Q^1) \Big) \in E(G)$ and $K(\mathcal{S}_1, Q^1) \in F\Big(K(\mathcal{S}_1, Q^1) \Big)$. For $n = p + 1$, we define

$$B^n(\mathcal{S}_1, Q^1) = \Big(B_1^n(\mathcal{S}_1, Q^1), \ldots, B_p^n(\mathcal{S}_1, Q^1) \Big) \in F\Big(B^p(\mathcal{S}_1, Q^1) \Big)$$

by

$$B_i^n(\mathcal{S}_1, Q^1) = \begin{cases} \emptyset, & \text{if } i \notin \bigcup_{C \in \mathcal{S}_1} [C]_{\leftarrow}, \\ U_i^e\Big(B^p(\mathcal{S}_1, Q^1), Q_i^1 \Big), & \text{if } i \in V^e \cap \bigcup_{C \in \mathcal{S}_1} [C]_{\leftarrow}, \\ W_i\Big(B^p(\mathcal{S}_1, Q^1) \Big) & \\ \quad \cup U_i^c\Big(B^p(\mathcal{S}_1, Q^1), Q_{\widehat{C}}^1 \Big), & \text{if } \widehat{C} \in C(H) \text{ and} \\ & i \in V(\widehat{C}) \cap \bigcup_{C \in \mathcal{S}_1} [C]_{\leftarrow}, \end{cases} \tag{3.7.5}$$

Since $B^p(\mathcal{S}_1, Q^1) \subset K(\mathcal{S}_2, Q^2)$, $K(\mathcal{S}_2, Q^2) \in F\Big(K(\mathcal{S}_2, Q^2) \Big)$, $Q^1 \subset Q^2$ and using the definitions of F and $K(\mathcal{S}_2, Q^2)$, we deduce that $B^{p+1}(\mathcal{S}_1, Q^1) \subset K(\mathcal{S}_2, Q^2)$. Also, $\Big(K(\mathcal{S}_1, Q^1), B^{p+1}(\mathcal{S}_1, Q^1) \Big) \in E(G)$. Arguing as in the proof of Proposition 3.4.1, one has

$$d\Big(K(\mathcal{S}_1, Q^1), B^{p+1}(\mathcal{S}_1, Q^1) \Big) \leq \lambda d\Big(K(\mathcal{S}_1, Q^1), B^p(\mathcal{S}_1, Q^1) \Big).$$

Repeating this argument, we obtain for every $n \geq p$, $B^n(\mathcal{S}_1, Q^1) \in K(\mathcal{S}_2, Q^2)$ such that $B^n(\mathcal{S}_1, Q^1) \to K(\mathcal{S}_1, Q^1)$. Therefore,

$$K(\mathcal{S}_1, Q^1) \subset K(\mathcal{S}_2, Q^2).$$

(5) Let $\mathcal{S} \subset C(H)$ and $\widehat{C} \in C(H)$ be such that $V(\widehat{C}) \subset \bigcup_{C \in \mathcal{S}} [C]_{\leftarrow}$. Let

$$Q = (Q_i)_{i \in V^e} \times (Q_C)_{C \in C(H)} \in \mathcal{Q}(\mathcal{S}).$$

We define

$$\widehat{Q} = (\widehat{Q}_i)_{i \in V^e} \times (\widehat{Q}_C)_{C \in C(H)}$$

by

$$\widehat{Q}_i = \begin{cases} Q_i, & \text{if } i \in V^e \text{ and } \mathcal{E}_i(\widehat{C}) \neq \emptyset, \\ \emptyset, & \text{if } i \in V^e \text{ and } \mathcal{E}_i(\widehat{C}) = \emptyset; \end{cases}$$
$$\widehat{Q}_C = \emptyset, \qquad \text{for } \; C \in C(H).$$

Clearly, $\widehat{Q} \in \mathcal{Q}(\widehat{C})$ and $\widehat{Q} \subset Q$. It follows from (2), (4) and Theorem 3.5.1(4) that

$$K(\widehat{C}, \widehat{Q}) \subset K(\mathcal{S}, Q)$$

and

$$K_i(\widehat{C}, \widehat{Q}) = K_i^+(\widehat{C}) = K_i^- \quad \forall i \in V(\widehat{C}).$$

□

Example 3.7.1. *Let $\{T_{i,j}\}_H$ be the H-IFS considered in Example 3.5.1. One has $C(H) = \{C_1, C_2\}$, $V^e = \emptyset$, $\mathcal{E}_{C_2}(C_2) = \emptyset$ and $\mathcal{E}_{C_1}(C_2) = \{(5,1), (5,2)\}$. For $k = 1, 2$ let $Q^k = Q^k_{C_1} \times Q^k_{C_2} \in \mathcal{Q}(C_2)$ be given by*

$$Q^1_{C_1} = \{(5,1)\}, \quad Q^2_{C_1} = \{(5,2)\}, \quad \textit{and} \quad Q^1_{C_2} = Q^2_{C_2} = \emptyset.$$

Figures 3.6 and 3.7 present $K(C_2, Q^1)$ and $K(C_2, Q^2)$ respectively. Observe that

$$K(C_2, Q^1) \neq K(C_2, Q^2), \quad K(C_2, Q^1) \subsetneq K^+(C_2) \quad \textit{and} \quad K(C_2, Q^2) \subsetneq K^+(C_2),$$

where $K^+(C_2)$ is presented in Figure 3.4.

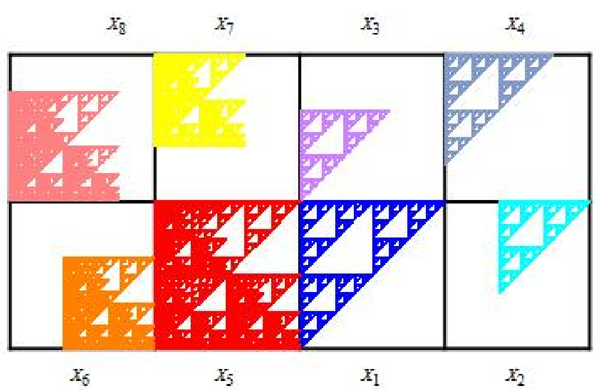

FIG. 3.6. The set $K(C_2, Q^1)$.

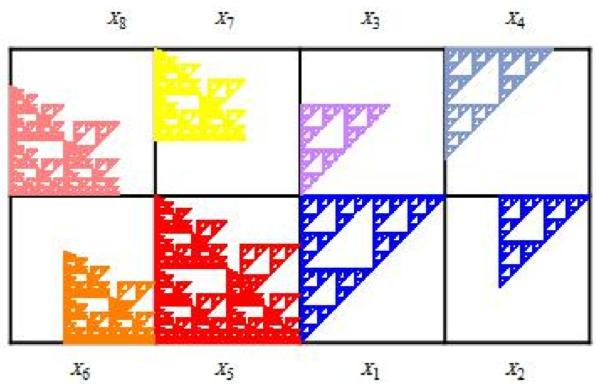

Fig. 3.7. The set $K(C_2, Q^2)$.

Chapter 4

A CONTRACTION PRINCIPLE ON GAUGE SPACES WITH GRAPHS AND APPLICATIONS TO INFINITE GRAPH-DIRECTED ITERATED FUNCTION SYSTEMS

4.1. Introduction

In 2008, Jachymski [29] introduced the notion of single-valued G-contraction defined on a complete metric space endowed with a graph, which is a map preserving the graph and satisfying a contraction condition only between points related by an edge. He proved some generalizations of the Banach contraction principle to single-valued G-contractions. In particular, he generalized many contractions results in partially ordered sets, see [35, 36, 43, 44].

In [10], Dinevari and Frigon generalized Jachymski's fixed point results to multi-valued maps by introducing the notions of multi-valued G-contraction and weak G-contraction on a complete metric space endowed with a graph. Other generalizations of Jachymski's results to multi-valued maps were obtained in [34].

In 1982, Gheorghiu [22] presented a fixed point result for general single-valued contractions in complete gauge spaces. In [5], Chiş and Precup extended his result and they presented a continuation principle for such contractions. Another approach to obtain fixed point results was developed in [19] for single-valued contractions and in [20] for multi-valued contractions on complete gauge spaces, (see also [21] for a survey of results on that subject).

In this paper, we consider a complete gauge space X endowed with a directed graph G. We introduce the notions of multi-valued G-contraction and G-Lipschitz multi-valued map in the sense of Gheorghiu on X. Then, we establish a fixed point result for such multi-valued maps. This result generalizes fixed point results for single-valued and multi-valued contractions on complete metric spaces endowed with a graph obtained in [29] and [10] respectively. It is worthwhile to notice that our fixed point result is new even in the particular case where the map is a single-valued and defined on X.

In this paper, we are also interested to apply our fixed point result to infinite iterated function systems.

An iterated function system (IFS) is a finite set of self-maps $\{T_i : i = 1, \dots, n\}$ defined on a complete metric space (M, d). Using the Banach contraction principle, Hutchinson [28] proved that if each T_i is a contraction, then there exists a unique nonempty compact set $K \subset M$, called the attractor of the IFS, such that

$$K = \bigcup_{i=1}^{n} T_i(K).$$

This result was popularized by Barnsley [2] as the main method of constructing fractals.

Geometric graph-directed constructions are generalizations of iterated function systems. Mauldin and Williams [31] were the firsts who introduced the notion of graph-directed constructions in $\mathbb{R}^m$ governed by a finite directed graph H and the similarity maps $T_{i,j}$ which are labeled with the edges of the graph. They established that each geometric graph-directed construction has a unique attractor. Graph-directed constructions have been studied and generalized by many authors, see for exemple [8, 14, 25] and the references therein.

Recently, Dinevari and Frigon [11] applied their fixed point results for multi-valued G-contractions established in [10] to obtain more information on the attractor K of a graph-directed iterated function system governed by a finite directed graph and a finite family of contractions $\{T_{i,j}\}$ defined on complete metric spaces and labeled by the edges of the graph. To this aim, they defined a complete metric space, a suitable directed graph G on this space, and an appropriate multi-valued G-contraction. Using the fixed points of this G-contraction, they studied certain subsets of the attractor K and the relations between these sub-attractors.

In this paper, we consider a directed graph $H = (V(H), E(H))$ such that $V(H)$ the set of vertices and $E(H)$ the set of edges are countably infinite sets. We study infinite graph-directed iterated function systems over the graph H (H-IIFS). Such an H-IIFS contains a family of contractions $\{T_{i,j}\}_{(i,j)\in E(H)}$ on complete metric spaces. We give conditions insuring the existence of a unique attractor to this H-IIFS. Our result relies on a generalization of Gheorghiu's fixed point theorem on gauge spaces due to Chiş and Precup [5].

Then, under an extra assumption on the H-IIFS, we apply our fixed point result for multi-valued contractions on complete gauge spaces endowed with graphs to obtain more information on the attractor of this H-IIFS. Those results are obtained in Section 6. In order to prove those results, taking into account the H-IIFS, we construct a suitable complete gauge space on which we define an appropriate directed graph G in Section 4. In Section 5, we define a multi-valued map on this gauge space and we show that it is a G-contraction.

4.2. Infinite H-iterated function systems

In this section, we introduce the notions of infinite MW-graph H and infinite graph iterated function system over the graph H. We give conditions insuring the existence of a unique attractor to an infinite graph iterated function system over the graph H.

Definition 4.2.1. *A directed graph $H = (V(H), E(H))$ is called an infinite MW-directed graph if*

(i) $V(H)$ is countable;

(ii) H has no parallel edges;

(iii) $1 \leq outdeg(i) < \infty$ for every $i \in V(H)$, where $outdeg(i)$ is the number of outward directed edges emanating from vertex i.

Definition 4.2.2. *Let $H = (V(H), E(H))$ be an infinite MW-directed graph. An infinite graph iterated function system over the graph H (H-IIFS) is a family of nonempty complete metric spaces, $\{M_i : i \in V(H)\}$, and, for each $(i,j) \in E(H)$, a single-valued contraction $T_{i,j} : M_j \to M_i$ with constant of contraction $\lambda_{i,j}$. An H-IIFS is denoted by $\{T_{i,j}\}_H$.*

An attractor of an H-IIFS is defined as follows.

Definition 4.2.3. *Let $\{T_{i,j}\}_H$ be an H-IIFS. An attractor K of this H-IIFS is a family of nonempty compact sets $K = (K_i)_{i\in V(H)}$ such that $K_i \subset M_i$ and*

$$K_i = \bigcup_{(i,j)\in E(H)} T_{i,j}(K_j) \quad \forall i \in V(H).$$

In order to establish the existence of an attractor to some H-IIFS, we will use the following generalization of Gheorghiu's fixed point result due to Chiş and Precup [5] that we recall for sake of completeness.

Theorem 4.2.1 ([5])**.** *Let $(X, \{q_s\}_{s\in S})$ be a complete gauge space, and $f : X \to X$ a single-valued map. Assume that*

(i) there exist a function $\psi : S \to S$ and $k = (k_s)_{s\in S}$ such that $k_s \geq 0$ for all $s \in S$,

$$q_s\Big(f(x), f(y)\Big) \leq k_s q_{\psi(s)}(x,y) \quad \forall s \in S,\ \forall x,y \in X, \tag{4.2.1}$$

and

$$\sum_{n=1}^{\infty} k_s k_{\psi(s)} \cdots k_{\psi^{n-1}(s)} q_{\psi^n(s)}(x,y) < \infty \quad \forall s \in S,\ \forall x,y \in X,$$

where ψ^n is the n-th iteration of ψ;

(ii) for every $x_0 \in X$, if $\{f^n(x_0)\}$ converges to some $x \in X$, then $x = f(x)$. Then f has a unique fixed point.

We need to introduce some notations. In what follows, H is an infinite MW-directed graph and $\{T_{i,j}\}_H$ is an H-IIFS.

Let

$$\Gamma_0 = \Big\{ I = \{i_1, \ldots, i_n\} \subset V(H) : n \in \mathbb{N} \Big\}. \tag{4.2.2}$$

We denote

$$k_I = \max \Big\{ \lambda_{i,j} : (i,j) \in E(H) \text{ and } i \in I \Big\} \quad \forall I \in \Gamma_0,$$

and we define the map $\varphi : \Gamma_0 \to \Gamma_0$ by

$$\varphi(I) = I \cup \Big\{ j \in V(H) : \exists i \in I \text{ such that } (i,j) \in E(H) \Big\}. \tag{4.2.3}$$

We consider the space

$$\mathcal{Y} = \Big\{ Y = (Y_i)_{i \in V(H)} : \emptyset \neq Y_i \subset M_i \text{ is compact} \Big\}. \tag{4.2.4}$$

For every $I \in \Gamma_0$ and $Y, \hat{Y} \in \mathcal{Y}$, let

$$p_I(Y, \hat{Y}) = \max \Big\{ D_i(Y_i, \hat{Y}_i) : i \in I \Big\}, \tag{4.2.5}$$

where D_i is the Hausdorff metric on M_i. It is easy to see that $(\mathcal{Y}, \{p_I\}_{I \in \Gamma_0})$ is a complete gauge space.

We are ready to establish the existence of an attractor of the H-IIFS.

Theorem 4.2.2. *Let $\{T_{i,j}\}_H$ be an H-IIFS. Assume that*

$$\sum_{n=1}^{\infty} k_I k_{\varphi(I)} \cdots k_{\varphi^{n-1}(I)} p_{\varphi^n(I)}(Y, \hat{Y}) < \infty \quad \forall I \in \Gamma_0,\ \forall Y, \hat{Y} \in \mathcal{Y}. \tag{4.2.6}$$

Then $\{T_{i,j}\}_H$ has a unique attractor K.

Proof. Let us define $f : \mathcal{Y} \to \mathcal{Y}$ by

$$f_i(Y) = \bigcup_{(i,j) \in E(H)} T_{i,j}(Y_j).$$

Using the fact that every $T_{i,j}$ is a contraction in the classical sense, we prove that

$$p_I(f(Y), f(\hat{Y})) \leq k_I p_{\varphi(I)}(Y, \hat{Y}) \quad \forall I \in \Gamma_0,\ \forall Y, \hat{Y} \in \mathcal{Y}.$$

Indeed,

$$
\begin{aligned}
p_I(f(Y), f(\hat{Y})) &= \max\Big\{D_i(f_i(Y), f_i(\hat{Y})) : i \in I\Big\} \\
&= \max\left\{D_i\Big(\bigcup_{(i,j)\in E(H)} T_{i,j}(Y_j), \bigcup_{(i,j)\in E(H)} T_{i,j}(\hat{Y}_j)\Big) : i \in I\right\} \\
&\le \max\left\{\max_{(i,j)\in E(H)} D_i(T_{i,j}(Y_j), T_{i,j}(\hat{Y}_j)) : i \in I\right\} \\
&\le \max\left\{\max_{(i,j)\in E(H)} \lambda_{i,j} D_j(Y_j, \hat{Y}_j) : i \in I\right\} \\
&\le k_I \max\Big\{D_i(Y_i, \hat{Y}_i) : i \in \varphi(I)\Big\} \\
&= k_I p_{\varphi(I)}(Y, \hat{Y}).
\end{aligned}
$$

We claim that (ii) of Theorem 4.2.1 is satisfied. Indeed, let us assume that $Y^0 \in \mathcal{Y}$ is such that $\{f^n(Y^0)\}$ converges to some $Y \in \mathcal{Y}$. If $Y \neq f(Y)$, there exists $i \in V(H)$ such that

$$D_i(Y_i, f(Y)_i) = r > 0.$$

Let $N \in \mathbb{N}$ be such that

$$p_{\varphi(\{i\})}\Big(f^n(Y^0), Y\Big) < \frac{r}{2} \quad \forall n \ge N.$$

So,

$$
\begin{aligned}
r = p_{\{i\}}(Y, f(Y)) &\le p_{\{i\}}\Big(Y, f^{N+1}(Y^0)\Big) + p_{\{i\}}\Big(f^{N+1}(Y^0), f(Y)\Big) \\
&\le p_{\varphi(\{i\})}\Big(Y, f^{N+1}(Y^0)\Big) + k_{\{i\}} p_{\varphi(\{i\})}\Big(f^N(Y^0), Y\Big) < r.
\end{aligned}
$$

Contradiction.

It follows from Theorem 4.2.1 that f has a unique fixed point $K \in \mathcal{Y}$, and hence, K is an attractor of $\{T_{i,j}\}_H$. □

Remark 4.2.1. *Observe that* (4.2.6) *is satisfied if:*

$$\sup\{\lambda_{i,j} : (i,j) \in E(H)\} < 1 \quad \textit{and} \quad \sup\{\textit{diam}(M_i) : i \in V(H)\} < \infty. \tag{4.2.7}$$

So, every H-IIFS satisfying (4.2.7) *has a unique attractor.*

4.3. Multi-valued contractions on gauge spaces endowed with a graph

In this section, we consider $\Big(X, \{q_s\}_{s\in S}\Big)$ a complete gauge space endowed with a directed graph $G = (V(G), E(G))$ such that the set of vertices $V(G) = X$ and the set of edges $E(G)$ has no parallel edges and it contains the diagonal. We generalize Theorem 4.2.1 to multi-valued map $F : X \to X$ satisfying a condition analogous to (4.2.1) only for $x, y \in X$ related by an edge $(x, y) \in E(G)$.

Definition 4.3.1. *Let $F : X \to X$ be a multi-valued map with nonempty values. We say that F is a G-Lipschitz map in the sense of Gheorghiu with map $\psi : S \to S$ and constant $\lambda = (\lambda_s)_{s \in S}$ such that $\lambda_s \geq 0$ for all $s \in S$, if, for every $(x, y) \in E(G)$ and every $u \in F(x)$, there exists $v \in F(y)$ such that $(u, v) \in E(G)$ and*

$$q_s(u, v) \leq \lambda_s q_{\psi(s)}(x, y) \quad \forall s \in S. \tag{4.3.1}$$

The map F is called a G-contraction if it is a G-Lipschitz map with $\lambda_s < 1$ for every $s \in S$.

We consider suitable trajectories in X.

Definition 4.3.2. *Let $F : X \to X$ be a multi-valued mapping and $x_0 \in X$. We say that a sequence $\{x_n\}$ is a G-Picard trajectory from x_0, if $x_n \in F(x_{n-1})$ and $(x_{n-1}, x_n) \in E(G)$ for all $n \in \mathbb{N}$. The set of all such G-Picard trajectories from x_0 is denoted by $T(F, G, x_0)$.*

Here is our main fixed point result for multi-valued contractions in the sense of Gheorghiu on the gauge space X endowed with a directed graph G.

Theorem 4.3.1. *Let $F : X \to X$ be a multi-valued G-Lipschitz map with constant $\lambda = (\lambda_s)_{s \in S}$ and map $\psi : S \to S$. Assume that there exists $(x_0, x_1) \in E(G)$ such that $x_1 \in F(x_0)$ and*

$$\sum_{n=1}^{\infty} \lambda_s \lambda_{\psi(s)} \cdots \lambda_{\psi^{(n-1)}(s)} q_{\psi^n(s)}(x_0, x_1) < \infty \quad \forall s \in S. \tag{4.3.2}$$

Then, there exists a G-Picard trajectory from x_0 converging to some $\hat{x} \in X$. In addition, assume that one of the following conditions holds:

(i) *F is G*-Picard continuous from *x_0, i.e. the limit of any convergent G-Picard trajectory $\{x_n\} \in T(F, G, x_0)$ is a fixed point of F;*

(ii) *F has closed values and, for every $\{x_n\}$ in $T(F, G, x_0)$ converging to some $x \in X$, there exists a subsequence $\{x_{n_k}\}$ such that $(x_{n_k}, x) \in E(G)$ for all $k \in \mathbb{N}$.*

Then, $\hat{x}$ is a fixed point of F. Moreover, every converging G-Picard trajectory from x_0 converges to a fixed point of F.

Proof. Let x_0 and $x_1 \in F(x_0)$ be given by assumption. Since F is a G-Lipschitz map, one can choose a sequence $\{x_n\}$ such that $x_{n+1} \in F(x_n)$, $(x_n, x_{n+1}) \in E(G)$ and

$$q_s(x_n, x_{n+1}) \leq \lambda_s q_{\psi(s)}(x_{n-1}, x_n) \leq \ \ldots \ \leq \lambda_s \lambda_{\psi(s)} \ldots \lambda_{\psi^{n-1}(s)} q_{\psi^n(s)}(x_0, x_1),$$

for every $s \in S$ and $n \in \mathbb{N}$. Moreover, for every $m \in \mathbb{N}$,

$$q_s(x_n, x_{n+m}) \leq \sum_{i=n}^{n+m-1} q_s(x_i, x_{i+1}) \leq \sum_{i=n}^{n+m-1} \lambda_s \lambda_{\psi(s)} \ldots \lambda_{\psi^{i-1}(s)} q_{\psi^i(s)}(x_0, x_1).$$

Therefore, $\{x_n\}$ is a Cauchy sequence and hence converges to some $\hat{x} \in X$.

If the condition (i) is satisfied, then clearly $\hat{x}$ is a fixed point of F.

On the other hand, if the condition (ii) is satisfied, then there exists a subsequence $\{x_{n_k}\}$ such that $(x_{n_k}, \hat{x}) \in E(G)$ for every $k \in \mathbb{N}$. Since F is a G-Lipschitz map, for each $k \in \mathbb{N}$, there exists $y_{n_k+1} \in F(\hat{x})$ such that $(x_{n_k+1}, y_{n_k+1}) \in E(G)$ and

$$q_s(x_{n_k+1}, y_{n_k+1}) \leq \lambda_s q_{\psi(s)}(x_{n_k}, \hat{x}) \quad \forall s \in S.$$

Therefore, for every $s \in S$,

$$q_s(y_{n_k+1}, \hat{x}) \leq q_s(y_{n_k+1}, x_{n_k+1}) + q_s(x_{n_k+1}, \hat{x}) \leq \lambda_s q_{\psi(s)}(x_{n_k}, \hat{x}) + q_s(x_{n_k+1}, \hat{x}).$$

Consequently, $y_{n_k+1} \to \hat{x}$, and hence $\hat{x} \in F(\hat{x})$ since F has closed values. $\square$

Remark 4.3.1. *We could have formulated a more general result by considering two families of gauges as it is done in [6, 22]. We preferred not to do so for sake a simplicity.*

In the particular case where X is a metric space, the previous result generalizes a fixed point result for multi-valued contraction obtained in [10]. If, in addition F is single-valued, the fixed point result for G-contraction due to Jachymski [29] is generalized by the following result.

Corollary 4.3.1. *Let $f : X \to X$ be a single-valued map such that there exist $\psi : S \to S$ and $\lambda = (\lambda_s)_{s \in S}$ such that $\lambda_s \geq 0$ for all $s \in S$, and for every $(x, y) \in E(G)$*

$$(f(x), f(y)) \in E(G) \quad \textit{and} \quad q_s(f(x), f(y)) \leq \lambda_s q_{\psi(s)}(x, y) \quad \forall s \in S. \tag{4.3.3}$$

Assume that there exists $x_0 \in X$ such that $(x_0, f(x_0)) \in E(G)$ and

$$\sum_{n=1}^{\infty} \lambda_s \lambda_{\psi(s)} \cdots \lambda_{\psi^{(n-1)}(s)} q_{\psi^n(s)}(x_0, f(x_0)) < \infty \quad \forall s \in S. \tag{4.3.4}$$

Then, the sequence $\{f^n(x_0)\}$ converges to some $\hat{x} \in X$. In addition, assume that one of the following conditions holds:

(i) $f\Big(f^n(x_0)\Big) \to f(\hat{x})$;

(ii) there exists a subsequence $\{f^{n_k}(x_0)\}$ such that $(f^{n_k}(x_0), \hat{x}) \in E(G)$ for all $k \in \mathbb{N}$.

Then, $\hat{x}$ is a fixed point of f.

It is worthwhile to point out that in Theorem 4.3.1, we did not assume the continuity of the G-Lipschitz map F. The following lemma could be useful to deduce that the limit of a convergent G-Picard trajectory is a fixed point of F.

Lemma 4.3.1. *Let $F : X \to X$ be a multi-valued G-Lipschitz map with constant $\lambda = (\lambda_s)_{s\in S}$ and map $\psi : S \to S$. Assume that there exists $x_0 \in X$ and a G-Picard trajectory $\{x_n\}$ from x_0 converging to some $\hat{x} \in X$. In addition, assume that there exists $\hat{u} \in F(\hat{x})$ such that, for every $s \in S$, the following conditions hold:*

(i) there exists a subsequence $\{x_{n_k}\}$ such that there exists $\{\hat{x}_{n_k}\}$ a sequence in X satisfying

$$(\hat{x}, \hat{x}_{n_k}) \in E(G)\ \forall k \in \mathbb{N}, \quad \text{and} \quad q_{\psi(s)}(x_{n_k}, \hat{x}_{n_k}) \to 0;$$

(ii) for every $k \in \mathbb{N}$, one can choose $u_{n_k} \in F(\hat{x}_{n_k})$ such that

$$(\hat{u}, u_{n_k}) \in E(G) \quad \text{and} \quad q_s(\hat{u}, u_{n_k}) \leq \lambda_s q_{\psi(s)}(\hat{x}, \hat{x}_{n_k}),$$

satisfying

$$q_s(u_{n_k}, x_{n_k+1}) \to 0 \quad \text{as } k \to \infty.$$

Then, $\hat{x} = \hat{u} \in F(\hat{x})$.

Proof. Let us suppose that $\hat{x} \neq \hat{u}$. Then, there exists $s \in S$ such that

$$q_s(\hat{u}, \hat{x}) = r > 0.$$

Observe that

$$\begin{aligned} q_s(\hat{u}, \hat{x}) &\leq q_s(\hat{u}, u_{n_k}) + q_s(u_{n_k}, x_{n_k+1}) + q_s(x_{n_k+1}, \hat{x}) \\ &\leq \lambda_s q_{\psi(s)}(\hat{x}, \hat{x}_{n_k}) + q_s(u_{n_k}, x_{n_k+1}) + q_s(x_{n_k+1}, \hat{x}) \\ &\leq \lambda_s q_{\psi(s)}(\hat{x}, x_{n_k}) + \lambda_s q_{\psi(s)}(x_{n_k}, \hat{x}_{n_k}) + q_s(u_{n_k}, x_{n_k+1}) + q_s(x_{n_k+1}, \hat{x}) \\ &\to 0. \end{aligned}$$

Contradiction. So, $\hat{x} = \hat{u} \in F(\hat{x})$. ☐

4.4. A suitable gauge space endowed with a directed graph

In order to get more information on the attractor to the H-IIFS, we will apply our main fixed point result for multi-valued G-contraction. In this section, we will define a suitable complete gauge space.

First, we need to introduce some notations. For a graph $H = (V(H), E(H))$, we denote an N-*directed path* in H from i_0 to i_N by $[i_n]_{n=0}^N$, and we denote the set of vertices from which there is a directed path in H reaching $i \in H$ by

$$[i]_{\leftarrow} = \{j \in V(H) : \text{ there is a directed path from } j \text{ to } i \text{ in } H\}. \tag{4.4.1}$$

We say that a subgraph $C = (V(C), E(C))$ of H is *connected* if for every $i, j \in V(C)$ there exists a directed path from i to j in C. A *connected component* of H is a maximal connected subgraph of H. A subgraph $C = (V(C), E(C))$ of H is *weakly connected* if the undirected graph induced by C is connected. Let C and $\widehat{C}$ be two connected components of H. We write

$$C \preceq \widehat{C} \quad \Longleftrightarrow \quad \text{there is a directed path from } C \text{ to } \widehat{C}.$$

Also, we write $C \prec \widehat{C}$ if $C \preceq \widehat{C}$ and $C \neq \widehat{C}$. We say that C and $\widehat{C}$ are *incomparable* if $C \npreceq \widehat{C}$ and $\widehat{C} \npreceq C$.

Let H be an infinite MW-directed graph and $\{T_{i,j}\}_H$ an H-IIFS with M_i a complete metric space for every $i \in V(H)$. We denote the set of all connected components of H by

$$C(H) = \{C : C \text{ is a connected component of } H\}. \tag{4.4.2}$$

In what follows, we will make the following assumption:

(H) H is an infinite MW-directed graph and $\{T_{i,j}\}_H$ is an H-IIFS such that

(H1) H is weakly connected and

$$V(H) = \bigcup_{C \in C(H)} V(C);$$

(H2) for every $i, j \in V(H)$, the length of directed paths from i to j is bounded, i.e.

$$\sup\Big\{N : \exists [i_n]_{n=0}^N \text{ from } i = i_0 \text{ to } j = i_N \text{ containing no cycle}\Big\} < \infty;$$

(H3) the metric spaces M_i are bounded and

$$R = \sup\{diam(M_i) : i \in V(H)\} < \infty.$$

It follows from Definition 4.2.1 that $C(H)$ is countable. Let

$$\Gamma = \Big\{I \subset V(H) : 0 < \mathrm{card}(I) < \infty, \text{ and } V(C) \subset I \ \forall C \in C(H) \text{ such that } V(C) \cap I \neq \emptyset\Big\}. \tag{4.4.3}$$

We define the map $\phi : \Gamma \to \Gamma$ by

$$\phi(I) = I \cup \Big\{k \in V(H) : \text{ there exist } (i,j) \in E(H) \text{ and } C \in C(H) \text{ such that } i \in I \text{ and } j, k \in V(C)\Big\}. \tag{4.4.4}$$

We are ready to define our suitable gauge space.

(X) Let $\mathcal{X}$ be the space of elements $X = (X_i)_{i\in V(H)}$ satisfying the following properties:

(X1) X_i is a compact subset of M_i for every $i \in V(H)$;

(X2) there exists $i \in V(H)$ such that $X_i \neq \emptyset$;

(X3) if $X_i \neq \emptyset$ for some $i \in V(C)$ and $C \in C(H)$, then $X_j \neq \emptyset$ for all $j \in V(C)$.

Taking into account the graph H, we endow $\mathcal{X}$ with a directed graph defined as follows.

(G) Let $G = (V(G), E(G))$ be the directed graph such that $V(G) = \mathcal{X}$ and, for $X, Y \in \mathcal{X}$, $(X, Y) \in E(G)$ if and only if, for every $i \in V(H)$, one of the following properties holds:

(Ga) $X_i = Y_i = \emptyset$, or $X_i \neq \emptyset$ and $Y_i \neq \emptyset$;

(Gb) $X_i = \emptyset$, $Y_i \neq \emptyset$ and, for $C \in C(H)$ such that $i \in V(C)$, there exist $k \in V(C)$ and $j \in V(H)\backslash V(C)$ such that $(k, j) \in E(H)$ and $X_j \neq \emptyset$.

We endow $\mathcal{X}$ with the family of gauges $\{d_I\}_{I\in\Gamma}$, where

$$d_I(X, Y) = \max\left\{\overline{D}_i(X_i, Y_i) : i \in I\right\}, \tag{4.4.5}$$

with

$$\overline{D}_i(X_i, Y_i) = \begin{cases} D_i(X_i, Y_i), & \text{if } X_i \neq \emptyset, Y_i \neq \emptyset, \\ 0, & \text{if } X_i = \emptyset = Y_i, \\ R_i, & \text{otherwise}, \end{cases} \tag{4.4.6}$$

where D_i the Hausdorff metric in M_i and

(R) the family of constants $(R_i)_{i\in V(H)}$ is such that

(R1) for every $i \in V(H)$, $R_i > R$;

(R2) for every $C \in C(H)$, $R_i = R_j$ for all $i, j \in V(C)$;

(R3) for every $i, j \in V(H)$, if $R_i < R_j$, then $j \notin [i]_{\leftarrow}$;

(R4) for every $I \in \Gamma$, one has $R_i < R_j$ for every $i \in I$ and $j \in \phi(I)\backslash I$.

It is clear that $(\mathcal{X}, \{d_I\}_{I\in\Gamma})$ is a complete gauge space.

Now, we show that we can easily find $(R_i)_{i\in V(H)}$ satisfying (R).

Lemma 4.4.1. *Let H be an infinite MW-directed graph and $\{T_{i,j}\}_H$ an H-IIFS satisfying* (H). *Then, there exists $\{V_\mu : \mu \in L\}$ a family of non empty disjoint subsets with $L \subset \mathbb{Z}$ countable such that*

(1) $V(H) = \bigcup_{\mu\in L} V_\mu$;

(2) for every $C \in C(H)$, if $V(C) \cap V_\mu \neq \emptyset$ for some $\mu \in L$, one has $V(C) \subset V_\mu$;

(3) for every $C, \widehat{C} \in C(H)$ such that $C \prec \widehat{C}$, $V(C) \subset V_\mu$ and $V(\widehat{C}) \subset V_\nu$, one has $\mu < \nu$;

(4) if $\mu < \nu$ in L, then $j \notin [i]_{\leftarrow}$ for all $i \in V_\mu$ and $j \in V_\nu$.
Moreover, for every strictly increasing map $\sigma : L \to]1, \infty[$, the family of constants $(R_i)_{i \in V(H)}$ defined by

$$R_i = \sigma(\mu) R \quad \text{if } i \in V_\mu,$$

satisfies (R).

Proof. Let $\mathcal{S}_0 \subset C(H)$ be such that $\{C : C \in \mathcal{S}_0\}$ is a maximal set of incomparable connected components of H. We denote

$$\mathcal{S}_0^+ = \left\{C \in C(H) : \exists \widehat{C} \in \mathcal{S}_0 \text{ such that } \widehat{C} \prec C\right\};$$
$$\mathcal{S}_0^- = \left\{C \in C(H) : \exists \widehat{C} \in \mathcal{S}_0 \text{ such that } C \prec \widehat{C}\right\}.$$

It follows from (H1) that $C(H) = \mathcal{S}_0 \cup \mathcal{S}_0^+ \cup \mathcal{S}_0^-$. We denote

$$\mathcal{S}_1 = \left\{C \in \mathcal{S}_0^+ : \nexists \widehat{C} \in \mathcal{S}_0^+ \text{ such that } \widehat{C} \prec C\right\},$$

and we define inductively for each $n \in \mathbb{N}$,

$$\mathcal{S}_{n+1} = \left\{C \in \mathcal{S}_0^+ \setminus \bigcup_{k=1}^{n} \mathcal{S}_k : \nexists \widehat{C} \in \mathcal{S}_0^+ \setminus \bigcup_{k=1}^{n} \mathcal{S}_k \text{ such that } \widehat{C} \prec C\right\}.$$

Similarly, we denote

$$\mathcal{S}_{-1} = \left\{C \in \mathcal{S}_0^- : \nexists \widehat{C} \in \mathcal{S}_0^- \text{ such that } C \prec \widehat{C}\right\},$$

and we define inductively for each $n \in \mathbb{N}$,

$$\mathcal{S}_{-(n+1)} = \left\{C \in \mathcal{S}_0^- \setminus \bigcup_{k=1}^{n} \mathcal{S}_{-k} : \nexists \widehat{C} \in \mathcal{S}_0^- \setminus \bigcup_{k=1}^{n} \mathcal{S}_{-k} \text{ such that } C \prec \widehat{C}\right\}.$$

Let $L = \{\mu \in \mathbb{Z} : \mathcal{S}_\mu \neq \emptyset\}$ endowed with the natural order. We define

$$V_\mu = \bigcup_{C \in \mathcal{S}_\mu} V(C) \quad \forall \mu \in L.$$

Therefore, by (H),

$$V(H) = \bigcup_{\mu \in L} V_\mu.$$

By construction, (2), (3) and (4) are satisfied.

Let $\sigma : L \to]1, \infty[$ be a strictly increasing map, and the family of constants $(R_i)_{i \in V(H)}$ defined by

$$R_i = \sigma(\mu) R \quad \text{for} \quad i \in V_\mu.$$

The property (R) follows directly from (1)–(4) and the fact that $\sigma(L) \subset]1, \infty[$. $\square$

4.5. A suitable G-contraction

We consider H an infinite MW-directed graph and $\{T_{i,j}\}_H$ an H-IIFS satisfying the condition (H). In this section, we will define an appropriate multi-valued G-contraction on $\mathcal{X}$, where $\mathcal{X}$ is the space endowed with the family of gauges $\{d_I\}_{I\in\Gamma}$ and endowed with the directed graph G defined in the previous section. This G-contraction will be used to get more information on the attractor of this infinite H-IIFS.

Let $X \in \mathcal{X}$. If $j \in V(H)$ is such that $X_j \neq \emptyset$, then $T_{i,j}(X_j) \neq \emptyset$ for all i such that $(i,j) \in E(H)$. So, it is important to distinguish all those edges. To this aim, we introduce the following notation. For $C \in C(H)$,

$$E_C(X) = \{(k,j) \in E(H) : k \in V(C), j \notin V(C), X_j \neq \emptyset\}. \tag{4.5.1}$$

Let us notice that the cardinality of $E_C(X)$ is finite since outdeg(i) is finite for every $i \in V(H)$.

For $C \in C(H)$ and $i, k \in V(C)$, we define $T_{i\to k} : M_k \to M_i$ by

$$T_{i\to k}(x) = \Big\{T_{i_0,i_1} \circ \cdots \circ T_{i_{N-1},i_N}(x) : [i_n]_{n=0}^N \in \{i \xrightarrow{C} k\}\Big\}, \tag{4.5.2}$$

where

$$\begin{aligned}\{i \xrightarrow{C} k\} = \{[i_n]_{n=0}^N &: [i_n]_{n=0}^N \text{ is an } N\text{-directed path in } C \\ &\text{from } i = i_0 \text{ to } k = i_N \text{ containing no cycle}\}.\end{aligned} \tag{4.5.3}$$

For $i \in V(C)$ with $C \in C(H)$, we define the following subsets of M_i:

$$O_i(X,P) = \begin{cases} \emptyset, & \text{if } P = \emptyset, \\ \displaystyle\bigcup_{(k,j)\in P} T_{i\to k} \circ T_{k,j}(X_j), & \text{if } \emptyset \neq P \subset E_C(X); \end{cases} \tag{4.5.4}$$

and

$$W_i(X) = \begin{cases} \emptyset, & \text{if } X_i = \emptyset, \\ \displaystyle\bigcup_{(i,j)\in E(C)} T_{i,j}(X_j), & \text{if } X_i \neq \emptyset, \end{cases} \tag{4.5.5}$$

where $E(C) = \{(k,j) \in E(H) : k, j \in V(C)\}$.

We have all the ingredients to introduce a suitable multi-valued map. We define $F : \mathcal{X} \to \mathcal{X}$ by

$$F(X) = \Big\{U = (U_i)_{i\in V(H)} \in \mathcal{X} : U_i \in F_i(X)\ \forall i \in V(H)\Big\}, \tag{4.5.6}$$

where, for $i \in V(C)$ for some $C \in C(H)$, $F_i(X)$ is defined as follows:

$$F_i(X) = \begin{cases} \emptyset, & \text{if } X_i = \emptyset \text{ and } E_C(X) = \emptyset, \\ \Big\{O_i(X,P) : \emptyset \neq P \subset E_C(X)\Big\}, & \text{if } X_i = \emptyset \text{ and } E_C(X) \neq \emptyset, \\ \Big\{W_i(X) \cup O_i(X,P) : P \subset E_C(X)\Big\}, & \text{if } X_i \neq \emptyset. \end{cases} \tag{4.5.7}$$

It is easy to see that F is well defined and has finite, and hence closed values.

We show that F is a multi-valued G-contraction.

Proposition 4.5.1. *Let H be an infinite MW-directed graph and $\{T_{i,j}\}_H$ an H-IIFS satisfying* (H)*. Let $(R_i)_{i\in V(H)}$ be a family of constants satisfying* (R)*. Then, the multi-valued map defined as above, $F : \mathcal{X} \to \mathcal{X}$ is a G-contraction.*

Proof. We show that F is a G-contraction with constant of contraction $\lambda = (\lambda_I)_{I\in\Gamma}$, where

$$\lambda_I = \max\left\{ \max\{\lambda_{i,j} : i \in I, (i,j) \in E(H)\}, \max\left\{\frac{R}{R_i} : i \in I\right\}, \max\left\{\frac{R_i}{R_j} : i \in I, j \in \phi(I)\backslash I\right\}\right\}, \tag{4.5.8}$$

where ϕ is defined in (4.4.4).

For $i, k \in V(C)$ for some $C \in C(H)$, we denote

$$\lambda_{i\to k} = \max\Big\{\lambda_{i_0,i_1}\cdots\lambda_{i_{N-1},i_N} : [i_n]_{n=0}^N \in \{i \xrightarrow{C} k\}\Big\}, \tag{4.5.9}$$

where $\{i \xrightarrow{C} k\}$ is given in (4.5.3). Observe that $\lambda_{i\to k} \le \lambda_I$ for all $I \in \Gamma$ such that $i \in I$.

Let $X, Y \in \mathcal{X}$ be such that $(X,Y) \in E(G)$ and $U \in F(X)$. We look for $\widetilde{U} \in F(Y)$ such that $(U,\widetilde{U}) \in E(G)$ and $d_I(U,\widetilde{U}) \le \lambda_I d_{\phi(I)}(X,Y)$ for every $I \in \Gamma$.

Step 1: For $\mathbf{I} \subset \mathbf{\Gamma}$, different cases of U_i for $\mathbf{i} \in \mathbf{I}$: Let $C \in C(H)$ be such that $i \in V(C) \subset I$.

Case 1: $U_i = \emptyset$ and $\widetilde{U}_i \neq \emptyset$ for every $\widetilde{U} \in F(Y)$.

In this case, $X_i = E_C(X) = \emptyset$ and $Y_i \cup E_C(Y) \neq \emptyset$ by (4.5.7).

If $Y_i \neq \emptyset$, since $(X,Y) \in E(G)$, by condition (Gb), there exist $k \in V(C)$ and $j \in V(H)\backslash V(C)$ such that $(k,j) \in E(H)$ and $X_j \neq \emptyset$. So, $(k,j) \in E_C(X)$. This contradicts the fact that $E_C(X) = \emptyset$.

If $E_C(Y) \neq \emptyset$, by (4.5.1), there exist $k \in V(C)$ and $j \in V(\widehat{C})$ such that $(k,j) \in E(H)$, $Y_j \neq \emptyset$ and $\widehat{C} \neq C$. One has $j \in \phi(I)\backslash I$ and $R_i < R_j$. Since

$E_C(X) = \emptyset$, one has $X_j = \emptyset$. By condition (Gb), there exist $m \in V(\widehat{C})$, $l \in V(H)\backslash V(\widehat{C})$ such that $(m,l) \in E(H)$ and $X_l \neq \emptyset$. So, $E_{\widehat{C}}(X) \neq \emptyset$ and $U_j \neq \emptyset$ by (4.5.7). So, we obtain

$$U_i = \emptyset,\ \widetilde{U}_i \neq \emptyset \quad \text{and} \quad U_j \neq \emptyset \text{ for some } (k,j) \in E_C(Y) \\ \text{with } k \in V(C) \text{ and } j \in \phi(I)\backslash I. \tag{4.5.10}$$

Moreover, by (4.4.5), (4.4.6) and (4.5.8),

$$\overline{D}_i(U_i, \widetilde{U}_i) = R_i = \frac{R_i}{R_j}\overline{D}_j(X_j, Y_j) \leq \lambda_I d_{\phi(I)}(X,Y) \quad \forall \widetilde{U} \in F(Y). \tag{4.5.11}$$

Case 2: $U_i \neq \emptyset$ and $\widetilde{U}_i = \emptyset$ for every $\widetilde{U} \in F(Y)$.

In this case, $X_i \cup E_C(X) \neq \emptyset$ and $Y_i \cup E_C(Y) = \emptyset$ by (4.5.7). Since $(X,Y) \in E(G)$, we deduce that $X_i = Y_i = \emptyset$ and hence $E_C(X) \neq \emptyset$. Let $(k,j) \in E_C(X)$. One has $X_j \neq \emptyset$ and $Y_j = \emptyset$, since $(k,j) \notin E_C(Y)$. This contradicts $(X,Y) \in E(G)$ (see condition (Ga)). Thus,

$$U_i \neq \emptyset \text{ and } \widetilde{U}_i = \emptyset \text{ for every } \widetilde{U} \in F(Y) \ \text{ is impossible.} \tag{4.5.12}$$

Case 3: $U_i \neq \emptyset$ and $\widetilde{U}_i \neq \emptyset$ for every $\widetilde{U} \in F(Y)$

In this case, $X_i \cup E_C(X) \neq \emptyset$ and $Y_i \cup E_C(Y) \neq \emptyset$ by (4.5.7).

If $X_i \neq \emptyset$, by condition (Ga), $Y_i \neq \emptyset$. So $W_i(X) \neq \emptyset$, $W_i(Y) \neq \emptyset$, and by (4.4.5), (4.5.5), and (4.5.8),

$$\begin{aligned} D_i(W_i(X), W_i(Y)) &= D_i\left(\bigcup_{(i,j)\in E(C)} T_{i,j}(X_j), \bigcup_{(i,j)\in E(C)} T_{i,j}(Y_j)\right) \\ &\leq \max_{(i,j)\in E(C)} D_i\Big(T_{i,j}(X_j), T_{i,j}(Y_j)\Big) \\ &\leq \max_{(i,j)\in E(C)} \lambda_{i,j} D_j(X_j, Y_j) \\ &\leq \lambda_I \max_{(i,j)\in E(C)} D_j(X_j, Y_j) \\ &\leq \lambda_I d_{\phi(I)}(X,Y). \end{aligned} \tag{4.5.13}$$

If $X_i = \emptyset$ and $Y_i \neq \emptyset$, then, for every $\widetilde{U}_i \in F_i(Y_i)$, one has by (4.4.6) and (4.5.8),

$$D_i(U_i, \widetilde{U}_i) \leq R = \frac{R}{R_i}\overline{D}_i(X_i, Y_i) \leq \lambda_I d_{\phi(I)}(X,Y). \tag{4.5.14}$$

If $E_C(X) \neq \emptyset$, for $\emptyset \neq P \subset E_C(X)$ such that $P \subset E_C(Y)$, for every $(k,j) \in P$, one has $j \in \phi(I)$, and, by (4.4.5), (4.5.2), (4.5.4), (4.5.8) and (4.5.9),

$$\begin{aligned} D_i(O_i(X,P), O_i(Y,P)) &= D_i\left(\bigcup_{(k,j)\in P} T_{i\to k} \circ T_{k,j}(X_j), \bigcup_{(k,j)\in P} T_{i\to k} \circ T_{k,j}(Y_j)\right) \\ &\leq \max_{(k,j)\in P} \lambda_{i\to k} D_k\Big(T_{k,j}(X_j), T_{k,j}(Y_j)\Big) \\ &\leq \max_{(k,j)\in P} \lambda_{i\to k}\lambda_{k,j} D_j(X_j, Y_j) \\ &\leq \lambda_I \max_{(k,j)\in P} D_j(X_j, Y_j) \\ &\leq \lambda_I d_{\phi(I)}(X,Y). \end{aligned} \tag{4.5.15}$$

If $P \subset E_C(X)$ and $P \not\subset E_C(Y)$, then there exists $(k,j) \in P$ such that $X_j \neq \emptyset$ and $Y_j = \emptyset$ which is impossible since $(X,Y) \in E(G)$.

Combining (4.5.7), (4.5.13), (4.5.14) and (4.5.15), we choose $\widetilde{U}_i \in F_i(Y)$ such that

$$\widetilde{U}_i = \begin{cases} W_i(Y), & \text{if } U_i = W_i(X), \\ O_i(Y,P), & \text{if } Y_i = \emptyset, \text{ and } U_i = O_i(X,P) \\ & \quad \text{for } \emptyset \neq P \subset E_C(X) \cap E_C(Y), \\ W_i(Y) \cup O_i(Y,P), & \text{if } Y_i \neq \emptyset, \text{ and} \\ & \quad U_i \in \{O_i(X,P), W_i(X) \cup O_i(X,P)\} \\ & \quad \text{for } \emptyset \neq P \subset E_C(X) \cap E_C(Y); \end{cases} \tag{4.5.16}$$

and we get

$$\overline{D}_i(U_i, \widetilde{U}_i) \leq \lambda_I d_{\phi(I)}(X,Y). \tag{4.5.17}$$

Step 2: Choice of an appropriate $\widetilde{\mathbf{U}} \in \mathbf{F}(\mathbf{Y})$:

Finally, we choose $\widetilde{U} = \left(\widetilde{U}_i\right)_{i\in V(H)} \in F(Y)$ as follows:

$$\widetilde{U}_i = \begin{cases} \emptyset, & \text{if } i \in V(C),\, U_i = \emptyset,\, Y_i \cup E_C(Y) = \emptyset, \\ \text{some } \widetilde{U}_i \in F_i(Y), & \text{if } i \in V(C),\, U_i = \emptyset,\, Y_i \cup E_C(Y) \neq \emptyset, \\ \widetilde{U}_i \text{ given by (4.5.16)}, & \text{if } i \in V(C),\, U_i \neq \emptyset,\, Y_i \cup E_C(Y) \neq \emptyset. \end{cases} \tag{4.5.18}$$

It follows from (4.5.10) and (4.5.16) that

$$(U, \widetilde{U}) \in E(G).$$

Finally, from (4.5.11) and (4.5.17), we deduce that

$$d_I(U, \widetilde{U}) \leq \lambda_I d_{\phi(I)}(X, Y) \quad \forall I \in \Gamma.$$

Therefore, F is a G-contraction. □

Remark 4.5.1. *From the proof of the previous proposition, we already know that for $(X, Y) \in E(G)$ and $U \in F(X)$, the choice of $\widetilde{U} \in F(Y)$ such that $(U, \widetilde{U}) \in E(G)$ and $d_I(U, \widetilde{U}) \leq \lambda_I d_{\phi(I)}(X, Y)$ for all $I \in \Gamma$ is not necessarily unique. Moreover, if for some $C \in C(H)$, one has $E_C(X) \neq \emptyset$, then, from the previous proof, we deduce that $E_C(X) \subset E_C(Y)$. So, for*

$$\emptyset \neq P \subsetneq \widetilde{P}, \quad \text{with } P \subset E_C(X), \widetilde{P} \subset E_C(Y), \tag{4.5.19}$$

there exists $(k, j) \in \widetilde{P} \backslash P$ with $X_j = \emptyset$ and $Y_j \neq \emptyset$. So, $j \in \phi(I) \backslash I$. By (4.4.5), (4.4.6) and (4.5.8),

$$\overline{D}_i(O_i(X, P), O_i(Y, \widetilde{P})) \leq R_i = \frac{R_i}{R_j} \overline{D}_j(X_j, Y_j) \leq \lambda_I d_{\phi(I)}(X, Y) \quad \forall i \in I.$$

Therefore, for $i \in V(C) \subset I$, $\widetilde{U}_i$ can be chosen as follows

$$\widetilde{U}_i = \begin{cases} W_i(Y), & \text{if } U_i = W_i(X), \\ O_i(Y, \widetilde{P}), & \text{if } Y_i = \emptyset \text{ and } U_i = O_i(X, P) \\ & \quad \text{with } \widetilde{P} \text{ as in (4.5.19)}, \\ W_i(Y) \cup O_i(Y, \widetilde{P}), & \text{if } Y_i \neq \emptyset, \text{ and} \\ & \quad U_i \in \{O_i(X, P), W_i(X) \cup O_i(X, P)\} \\ & \quad \text{with } \widetilde{P} \text{ as in (4.5.19)}. \end{cases}$$

4.6. Some properties of the attractor of an H-IIFS

For $H = (V(H), E(H))$ an infinite MW-directed graph, and $\{T_{i,j}\}_H$ an infinite graph-directed iterated function system over the graph H. Theorem 4.2.2 gave conditions insuring the existence of K an attractor of this H-IIFS. We want to get more information on K by taking into account the connected components of H. To this aim, we will consider $F : \mathcal{X} \to \mathcal{X}$ the G-contraction defined on the gauge space $\mathcal{X}$ endowed with the graph G introduced in sections 4 and 5.

Theorem 4.6.1. *Let $H = (V(H), E(H))$ be an infinite MW-directed graph and $\{T_{i,j}\}_H$ an H-IIFS satisfying* (H). *Let $(R_i)_{i\in V(H)}$ be a family of constants satisfying* (R). *Assume that $X^0 \in \mathcal{X}$ and $X^1 \in F(X^0)$ are such that*

$$\sum_{n=1}^{\infty} \lambda_I \lambda_{\phi(I)} \cdots \lambda_{\phi^{n-1}(I)} d_{\phi^n(I)}(X^0, X^1) < \infty \quad \forall I \in \Gamma, \tag{4.6.1}$$

where λ_I is defined in (4.5.8)*. Then, there exists $K(X^0) \in \mathcal{X}$ such that*

(1) $K_i(X^0) \neq \emptyset$ for every $i \in V(H)$ such that $X_i^0 \neq \emptyset$;
(2) $K_i(X^0) \neq \emptyset$ if and only if $i \in [j]_{\leftarrow}$, for some $j \in V(H)$ such that $X_j^0 \neq \emptyset$;
(3) $K(X^0)$ is a fixed point of the multi-valued map F;
(4) if $\{T_{i,j}\}_H$ has an attractor K, then $K(X^0) \subset K$.

Proof. Let $F : \mathcal{X} \to \mathcal{X}$ be the multi-valued map defined in (4.5.6) and (4.5.7). We know that F is a G-contraction by Proposition 4.5.1. Also, if $\{T_{i,j}\}_H$ has an attractor K, the definition of F implies that fixed points of F are included in K.

Let $X^0 \in \mathcal{X}$ and $X^1 \in F(X^0)$ be such that (4.6.1) is satisfied. We want to show that there exists $K(X^0)$ a fixed point of F satisfying the required properties.

For $n \in \mathbb{N}$, we choose inductively

$$X^{n+1} \in F(X^n) \quad \text{the biggest element of } F(X^n), \tag{4.6.2}$$

that is $X^{n+1} = (X_i^{n+1})_{i\in V(H)} \in F(X^n)$ is chosen as follows. For $i \in V(C)$ for some $C \in C(H)$,

$$X_i^{n+1} = \begin{cases} \emptyset, & \text{if } X_i^n = E_C(X^n) = \emptyset; \\ O_i\Big(X^n, E_C(X^n)\Big), & \text{if } X_i^n = \emptyset,\ E_C(X^n) \neq \emptyset; \\ W_i(X^n) \cup O_i\Big(X^n, E_C(X^n)\Big), & \text{if } X_i^n \neq \emptyset; \end{cases} \tag{4.6.3}$$

where E_C, O_i and W_i are defined in (4.5.1), (4.5.4) and (4.5.5) respectively.

Arguing as in the proof of Proposition 4.5.1 and by Remark 4.5.1, one has that $(X^{n-1}, X^n) \in E(G)$ and

$$d_I(X^n, X^{n+1}) \leq \lambda_I d_{\phi(I)}(X^{n-1}, X^n) \quad \forall I \in \Gamma.$$

By the proof of Theorem 4.3.1, the sequence $\{X^n\}$ is a G-Picard trajectory converging to some $K(X^0) \in \mathcal{X}$.

Observe that for every $i \in V(H)$ such that $X_i^0 \neq \emptyset$, one has $X_i^n \neq \emptyset$ for every $n \in \mathbb{N}$. Therefore, $K(X^0)$ satisfies (1).

By construction, for $i \in V(C)$ for $C \in C(H)$, if there is a directed path $[i_n]_{n=0}^N$ in H from $i = i_0$ to $j = i_N$ such that $X_j^0 \neq \emptyset$, then $X_i^n \neq \emptyset$ for every $n > N$. Therefore, $K(X^0)_i \neq \emptyset$. On the other hand, if $i \notin [j]_{\leftarrow}$, for all $j \in V(H)$ such that $X_j^0 \neq \emptyset$, then $X_i^n = \emptyset$ for every $n \in \mathbb{N}$, and hence $K(X^0)_i = \emptyset$. So, $K(X^0)$ satisfies (2).

To conclude, we have to show that $K(X^0)$ is a fixed point of F. This will imply that $K(X^0) \subset K$ if the attractor K of $\{T_{i,j}\}_H$ exists.

Let us denote

$$V(X^0) = \{i \in V(H) : i \in [j]_{\leftarrow} \text{ for some } j \in V(H) \text{ such that } X_j^0 \neq \emptyset\}. \tag{4.6.4}$$

It follows from (2) that

$$\begin{aligned} &\text{if } i \in V(X^0), \quad K(X^0)_i \neq \emptyset, \\ &\text{if } i \notin V(X^0), \quad K(X^0)_i = E_C(K(X^0)) = \emptyset. \end{aligned} \tag{4.6.5}$$

Let $\widehat{U} = (\widehat{U})_{i \in V(H)} \in \mathcal{X}$ be defined by

$$\widehat{U}_i = \begin{cases} \emptyset, & \text{if } i \in V(H) \backslash V(X^0), \\ W_i(K(X^0)) \cup O_i\Big(K(X^0), E_C(K(X^0))\Big), & \text{if } i \in V(X^0) \cap V(C) \\ & \quad \text{for } C \in C(H). \end{cases} \tag{4.6.6}$$

So, by (4.6.5) and the definition of F (see (4.5.7)),

$$\widehat{U} \in F(K(X^0)). \tag{4.6.7}$$

We claim that $K(X^0) = \widehat{U}$.

Let $\hat{I} \in \Gamma$. For every $C \in C(H)$ such that $V(C) \subset \hat{I}$, we denote

$$N_C = \begin{cases} \sup\Big\{ \inf\{n : X_j^n \neq \emptyset\} : (k,j) \in E_C(K(X^0))\Big\}, & \text{if } E_C(K(X^0)) \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

From the fact that $\operatorname{outdeg}(k) < \infty$ for every $k \in V(C)$ and by (H), we deduce that $N_C < \infty$. Let

$$N = \max\Big\{N_C : V(C) \subset \hat{I}\Big\}. \tag{4.6.8}$$

So,

$$E_C(K(X^0)) = E_C(X^n) \quad \forall V(C) \subset \hat{I}, \ \forall n > N. \tag{4.6.9}$$

For $n > N$, let us define $\widehat{X}^n = (\widehat{X}_i^n)_{i \in V(H)}$, $\widehat{U}^n = (\widehat{U}_i^n)_{i \in V(H)} \in \mathcal{X}$ by

$$\widehat{X}_i^n = \begin{cases} X_i^n, & \text{if } i \in \phi(\hat{I}), \\ K(X^0)_i, & \text{otherwise;} \end{cases}$$

and

$$\widehat{U}_i^n = \begin{cases} \emptyset, & \text{if } i \in V(H) \backslash V(X^0), \\ W_i(\widehat{X}^n) \cup O_i\Big(\widehat{X}^n, E_C(\widehat{X}^n)\Big), & \text{if } i \in V(X^0) \cap V(C) \text{ for } C \in C(H). \end{cases}$$

It follows from (4.6.9) and the definitions of $E(G)$ and F (see (4.5.6)) that

$$(K(X^0), \widehat{X}^n) \in E(G), \quad (\widehat{U}, \widehat{U}^n) \in E(G) \quad \text{and} \quad \widehat{U}^n \in F(\widehat{X}^n). \tag{4.6.10}$$

Arguing as in the proof of Proposition 4.5.1, we can show that

$$d_{\hat{I}}(\widehat{U}^n, \widehat{U}) \leq \lambda_{\hat{I}} d_{\phi(\hat{I})}(\widehat{X}^n, K(X^0)). \tag{4.6.11}$$

Observe that, for every $n > N$,

$$\widehat{X}_i^n = X_i^n \ \ \forall i \in \phi(\hat{I}) \quad \text{and} \quad \widehat{U}_i^n = X_i^{n+1} \ \ \forall i \in \hat{I}. \tag{4.6.12}$$

So,

$$d_{\phi(\hat{I})}(\widehat{X}^{N+k}, X^{N+k}) \to 0 \quad \text{and} \quad d_{\hat{I}}(\widehat{U}^{N+k}, X^{N+k+1}) \to 0 \quad \text{as } k \to \infty. \tag{4.6.13}$$

Combining (4.6.7), (4.6.10), (4.6.11), and (4.6.13), it follows from Lemma 4.3.1 that

$$K(X^0) = \widehat{U} \in F(K(X^0)).$$

□

Theorem 4.6.2. *Let $H = (V(H), E(H))$ be an infinite MW-directed graph and $\{T_{i,j}\}_H$ an H-IIFS satisfying* (H). *Let $(R_i)_{i \in V(H)}$ be a family of constants satisfying* (R). *Assume that, for $X^0, Y^0 \in \mathcal{X}$,* (4.6.1) *is satisfied with (X^0, X^1) and (Y^0, Y^1), where X^1 and Y^1 are the biggest elements of $F(X^0)$ and $F(Y^0)$ respectively. Then the following statements hold:*

(1) If X^0, Y^0 are such that $\{i \in V(H) : X_i^0 \neq \emptyset\} = \{i \in V(H) : Y_i^0 \neq \emptyset\}$ and $X_i^0 \subset Y_i^0$ for every $i \in V(H)$, then $K(X^0) = K(Y^0)$.

(2) If X^0, Y^0 are such that $\{i \in V(H) : X_i^0 \neq \emptyset\} \subset \{i \in V(H) : Y_i^0 \neq \emptyset\}$, then $K(X^0)_i \subset K(Y^0)_i$ for every $i \in V(H)$.

(3) If there is $N \in \mathbb{N}$ such that $\{i \in V(H) : X_i^0 \neq \emptyset\} \subset \{[j]_{\leftarrow}^N : Y_j^0 \neq \emptyset\}$, then $K(X^0)_i \subset K(Y^0)_i$ for every $i \in V(H)$, where $[j]_{\leftarrow}^N = \{k \in V(H) :$ there is a directed path $[i_n]_{n=0}^{N_k}$ in H from $k = i_0$ to $j = i_{N_k}$ with $N_k \leq N\}$.

Proof. (1) Let $\{X^n\}$ and $\{Y^n\}$ be the G-Picard trajectories defined inductively by (4.6.2) and such that $X^n \to K(X^0)$ and $Y^n \to K(Y^0)$. Observe that $(X^n, Y^n) \in E(G)$ for every $n \in \{0\} \cup \mathbb{N}$. Arguing as in the proof of Proposition 4.5.1, we deduce that

$$d_I(X^n, Y^n) \leq \lambda_I d_{\phi(I)}(X^{n-1}, Y^{n-1}) \quad \forall n \in \mathbb{N}, \ \forall I \in \Gamma.$$

Therefore, $\{X^n\}$ and $\{Y^n\}$ have the same limit; that is $K(X^0) = K(Y^0)$.

(2) Let $Z^0 = (Z_i^0)_{i \in V(H)} \in \mathcal{X}$ be defined by $Z_i^0 = X_i^0 \cup Y_i^0$. Let Z^1 be the biggest element of $F(Z^0)$. One can check that

$$\overline{D}_i(Z_i^0, Z_i^1) \leq \overline{D}_i(X_i^0, X_i^1) + \overline{D}_i(Y_i^0, Y_i^1) \quad \forall i \in V(H),$$

and hence

$$d_I(Z^0, Z^1) \leq d_I(X^0, X^1) + d_I(Y^0, Y^1) \quad \forall I \in \Gamma.$$

Thus, (Z^0, Z^1) satisfies (4.6.1). So, Y^0 and Z^0 verify the assumptions of (1). Therefore,

$$K(Y^0) = K(Z^0).$$

Let $\{X^n\}$ and $\{Z^n\}$ be the G-Picard trajectories defined inductively by (4.6.2) and such that $X^n \to K(X^0)$ and $Z^n \to K(Z^0)$. Since $X_i^0 \subset Z_i^0$, one has $X_i^n \subset Z_i^n$ for every $i \in V(H)$ and every $n \in \mathbb{N}$. Thus,

$$K(X^0)_i \subset K(Z^0)_i = K(Y^0)_i \quad \forall i \in V(H).$$

(3) Let $\{X^n\}$ and $\{Y^n\}$ be the G-Picard trajectories defined inductively by (4.6.2) and such that $X^n \to K(X^0)$ and $Y^n \to K(Y^0)$. The assumption implies that

$$\{i \in V(H) : X_i^0 \neq \emptyset\} \subset \{i \in V(H) : Y_i^N \neq \emptyset\}.$$

From the proof of Proposition 4.5.1,

$$d_I(Y^N, Y^{N+1}) \leq \lambda_I \cdots \lambda_{\phi^{N-1}(I)} d_{\phi^N(I)}(Y^0, Y^1) \quad \forall I \in \Gamma.$$

Therefore, (Y^N, Y^{N+1}) satisfies (4.6.1). It follows from (2) that

$$K(X^0)_i \subset K(Y^N)_i \quad \forall i \in V(H).$$

Since

$$K(Y^N) = \lim_{k \to \infty} Y^{N+k} = \lim_{n \to \infty} Y^n = K(Y^0),$$

one has

$$K(X^0)_i \subset K(Y^0)_i \quad \forall i \in V(H).$$

$\square$

CONCLUSION

The fixed point results for multivalued contractions defined on graphs have been extensively discussed within this thesis. Here we briefly mention again the specific achievements of this thesis. In Chapter 1, we established the fixed point results for multivalued contractions defined on complete metric spaces endowed with directed graphs. These results generalize and unify many of the previous results in this context. Besides developing and improving some of the fixed point results in the literature, we also illustrated the applications of our results to integral inclusions and fractals. In Chapter 2, we established existence results for a system of Hammerstein integral inclusions with mixed monotone conditions. In Chapter 3, we applied our fixed point results for multivalued contractions to obtain more information on the attractors of finite graph-directed iterated function systems. In Chapter 4, we established a fixed point result for multivalued contractions defined on complete gauge spaces endowed with graphs. We also applied this result to study the attractors of infinite graph-directed iterated function systems.

In this work, we developed different ideas but there are much more avenues that remained unexplored. In Chapter 1, we proved the fixed point results for G-contractions defined on a metric space X with a graph G. Much more has to be done in the particular case, where the G-contraction in not defined on X but only on a subset of X. In this case, extra assumptions are needed. One possibility would be to consider inward conditions to guaranty the existence of fixed points.

In Chapter 2, we considered a system of Hammerstein integral inclusions which includes N inclusions. Proving the existence results in the case when the system includes infinite number of inclusions has not been done yet and is very worthwhile doing.

In Chapter 4, in order to prove the existence of sub-attractors for infinite graph-directed iterated function systems, we assumed that the length of directed paths between each pair of vertices is bounded. Proving the existence result without this assumption is not an easy task and can be a good subject for further research.

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